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~~OPERATIONS~~ STOCHASTIC
~~MANAGEMENT~~ PROCESSES

The Coop

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Name

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The Poisson Process

Derivation
Time-dependent
Combination of independent P. processes
Multiple arrivals

A Queuing model as Birth & Death Process

Arrival rate λ_n
Departure rate μ_n

(*) $n \geq 1$: $\dot{P}_n(t) = \lambda_{n-1} P_{n-1}(t) - (\mu_n + \lambda_n) P_n(t) + \mu_{n+1} P_{n+1}(t)$

$$\dot{P}_0(t) = \mu_1 P_1(t) - \lambda_0 P_0(t)$$

Not soluble in general

Equilibrium distribution:

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_{n+1} = \frac{\lambda_n}{\mu_{n+1}} P_n + \left\{ \frac{\mu_n P_n - \lambda_{n-1} P_{n-1}}{\mu_{n+1}} \right\}$$

Using this expression for n rather than $n+1$ & substituting gives

$$P_{n+1} = \frac{\lambda_n}{\mu_{n+1}} P_n + \left\{ \frac{\mu_{n-1} P_{n-1} - \lambda_{n-2} P_{n-2}}{\mu_{n+1}} \right\} \quad \text{etc.}$$

Or: $\mu_n P_n - \lambda_{n-1} P_{n-1} = \mu_{n-1} P_{n-1} - \lambda_{n-2} P_{n-2} = \cdots =$

$$= \mu_1 P_1 - \lambda_0 P_0 = 0 .$$

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Hence: $P_{n+1} = \frac{\lambda_n}{\mu_{n+1}} P_n = \frac{\lambda_n \cdots \lambda_0}{\mu_{n+1} \cdots \mu_1} P_0$

Solve for P_0 by $1 = P_0 + P_0 \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$

Distribution of waiting time:

X = wtg time ; y = # in system at some random s.s. time.

$P_r\{X > 0\} = P_r\{y \geq c\} = \sum_{n=c}^{\infty} P_n = \sum_{n=c}^{\infty} P_c \rho^{n-c}$

for an ~~c~~-server system $\left\{ \begin{array}{l} \mu_n = \begin{cases} n\mu, & n \leq c \\ c\mu, & n > c \end{cases} \\ \lambda_n = \lambda \\ \rho \equiv \frac{\lambda}{\mu c} \end{array} \right.$

$W(x) \equiv P_r\{X \leq x\}$

$1 - W(x) = \sum_{n=c}^{\infty} P_r\{N(x) \leq n-c \mid y = n\} P_r\{y = n\}$

since waiting occurs only when all servers are busy

$1 - W(x) = \sum_{n=c}^{\infty} \left\{ \sum_{m=0}^{n-c} \frac{(c\mu x)^m e^{-c\mu x}}{m!} \right. P_c \rho^{n-c}$

$= P_c e^{-c\mu x} \sum_{m=0}^{\infty} \frac{(\rho c \mu x)^m}{m!} \sum_{n=c+m}^{\infty} \rho^{n-c-m}$

$= \frac{P_c}{1-\rho} e^{-(c\mu - \lambda)x}$

$W(x) = 1 - \frac{P_c}{1-\rho} e^{-(c\mu-\lambda)x}$

Example:

Crates arrive on platform at rate λ

Trucks (c) pick up at rate μ / truck

$\ell(x)$ = loss for each crate if in queue for time x

K = cost per hour of a truck

$$\text{cost} = \xi = Kc + \int_0^\infty \ell(x)\, dW(x)$$

$$= \ell(0) P_r\{x=0\} + \int_0^\infty \ell(x) P_r\{x>0\} e^{-(c\mu-\lambda)x} (c\mu-\lambda)\, dx \quad + Kc$$

Assume $\ell(0) = 0$ & $\min\limits_{c} \left\{ Kc + (c\mu-\lambda) P_r\{x>0\} \int_0^\infty \ell(x) e^{-(c\mu-\lambda)x} dx \right\}$

M/M/C with upper bound on capacity

$$\lambda_n = \begin{cases} \lambda, & 1 \le n \le B-1 \\ 0, & n \ge B \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & 0 < n \le c \\ c\mu & n > c \end{cases}$$

M/M/c/N : Finite population N

$$\lambda_n = \begin{cases} (N-n)\lambda, & n \le N \\ 0, & n > N \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & n \le c \\ c\mu, & n > c \end{cases}$$

M/M/1 ; state-dependent service rate

$\mu_n = n^p \mu$ where p is the pressure coefficient

$p > 0 \Leftrightarrow$ speed-up as queue increases
$p < 0 \Leftrightarrow$ slow-down " " "

M/M/c ; state-dependent arrival rate

$$\lambda_n = \begin{cases} \lambda_0 & \text{if } n \le c-1 \\ \left(\frac{c}{n+1}\right)^p \lambda_0 & \text{if } n \ge c \end{cases}$$

This can be interpreted as a balking model

$M/E_K/1$

$$t \sim \frac{\mu (\mu t)^{K-1} e^{-\mu t}}{(K-1)!}$$

We can treat this as a birth & death process if we redefine the state space (method of stages).

Let n = # of customers
s = stage of service , $s = 1, 2, \ldots, K$

$$\text{State} = \begin{cases} nK - s + 1 & n > 0 \\ 0 & n = 0 \end{cases}$$

service rate per stage is $K\mu$.

Renewal Theory

Defn: A renewal process is a sequence $T_1, T_2, \ldots$ of positive i.i.d. random variables where $F(\cdot)$ is the common distribution function with $F(0) = 0$.

$$W_n \equiv \sum_{i=1}^{n} T_i = \text{waiting time to } n^{th} \text{ event}$$

$$N(t) = n \iff W_n \le t < W_{n+1}$$

$\{N(t);\ t \ge 0\}$ is called a renewal counting process.

Example: A Poisson process is a renewal process

T_i are i.i.d. exponential with rate λ
W_n is gamma-distributed: $f_{W_n} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$, $t > 0$

Theorem: $N(t)$ has finite moments of all orders
Proof:

Assume $\exists\ c \ni 0 < F(c) < 1$

Let $x_i = \begin{cases} 0, & T_i < c \\ c, & T_i \ge c \end{cases}$ be a set of i.i.d. r.v.'s.

Define $W_m^x \equiv \sum_{i=1}^{m} x_i$ so that $N^x(t) = n \iff W_n^x \le t < W_{n+1}^x$

Since $x_i \le T_i$, $W_m^x \le W_m$ & $N(t) \le N^x(t)$

Thus we need only show that $N^x(t)$ has finite moments.
Now, $N^x(t) = \#$ of trials ~~before~~ with $\left[\frac{t}{c}\right] + 1$ of the T_i's are $\ge c$.

Hence $N^x(t)$ is negative-binomial-distributed & has finite moments.

<u>Theorem</u>: $m(t) \equiv E\{N(t)\}$ satisfies the relation

$$m(t) = F(t) + \int_0^t M(t-u)\,dF(u) \qquad \left(\begin{matrix}\text{The}\\ \text{Renewal}\\ \text{Equation}\end{matrix}\right)$$

$$= F(t) + M(t) * F(t)$$

Proof:

$$E\{N(t)\} = \int_0^\infty E\{N(t)\,|\,T_1 = \tau\}\,dF(\tau)$$

~~Now $E\{N(t)|T_1=\tau\} = \begin{cases} 0 & \text{if } \tau > t \\ 1 + N(t-\tau) & \text{if } \tau \le t\end{cases}$~~

$$\begin{aligned}
P_r\{N(t)=n\,|\,T_1=\tau\} &= 0 \quad \text{if } n=0 \\
&= P_r\{T_2+\cdots+T_n \le t-\tau < T_2+\cdots+T_{n+1}\,|\,T_1=\tau\} \\
&= \frac{P_r\{T_2+\cdots+T_n \le t-\tau\,;\, T_2+\cdots+T_{n+1} > t-\tau\,;\, T_1=\tau\}}{P_r\{T_1=\tau\}} \\
&= P_r\{T_2+\cdots+T_n \le t-\tau\,;\, T_2+\cdots+T_{n+1} < t-\tau\} \\
&= P_r\{N(t-\tau) = n-1\}
\end{aligned}$$

so that
$$\begin{aligned}
E\{N(t)\,|\,T_1=\tau\} &= \sum_{n=0}^{\infty} n\, P_r\{N(t)=n\,|\,T_1=\tau\} \\
&= \sum_{n=0}^{\infty} P_r\{N(t-\tau) = n-1\} \\
&= \sum_{k=0}^{\infty} (k+1)\, P_r\{N(t-\tau)=k\} \\
&= E\{N(t-\tau)\}+1 = 1 + M(t-\tau)
\end{aligned}$$

so
$$E\{N(t)\,|\,T_1=\tau\} = \begin{cases} 0 & \text{if } \tau > t \\ 1+M(t-\tau) & \text{if } \tau \le t \end{cases}$$

Hence
$$\begin{aligned}
M(t) &= \int_0^t [1+M(t-\tau)]\,dF(\tau) \\
&= F(t) + \int_0^t M(t-\tau)\,dF(\tau) \quad \longleftarrow
\end{aligned}$$

$\mathcal{F}(s) \equiv \int_0^\infty e^{-st}\,dF(t) \equiv$ Laplace-Stieltjes transform of $F(\cdot)$

$\mathcal{M}(s) = \mathcal{F}(s) + \mathcal{M}(s)\mathcal{F}(s)$

Knowing either $M(t)$ or $F(t)$ defines the other.

$$\mathcal{M}(s) = \frac{\mathcal{F}(s)}{1-\mathcal{F}(s)} \quad \text{or} \quad \mathcal{F}(s) = \frac{\mathcal{M}(s)}{1-\mathcal{M}(s)}$$

<u>Theorem</u>: If F_n is the distribution function of W_n, then

$$M(t) = \sum_{n=1}^{\infty} F_n(t)$$

Proof: $\sum_{n=1}^{\infty} F_n(t) = \sum_{n=1}^{\infty} \Pr\{W_n \le t\} = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \Pr\{N(t) = k\}$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{k} = \sum_{k=1}^{\infty} k \Pr\{N(t)=k\} = E\{N(t)\} = M(t)$$

$F_n(t) =$ n-fold convolution of $F(\cdot)$

or $\mathcal{M}(s) = \sum_{n=1}^{\infty} \mathcal{F}^n(s) = \mathcal{F}(s) \sum_{n=0}^{\infty} \mathcal{F}^n(s) = \frac{\mathcal{F}(s)}{1-\mathcal{F}(s)} = \mathcal{M}(s)$.

<u>Defn</u>: Present Age $= U(t) = t - W_{N(t)}$

Excess Life $= V(t) = W_{N(t)+1} - t$

<u>Theorem</u>: $\Pr\{V(t) \le x\} = F(t+x) - F(t) + \int_0^t [F(t+x-u) - F(t-u)]\, dM(u)$

Proof:

$$\Pr\{V(t) \le x\} = \sum_{n=0}^{\infty} \Pr\{V(t) \le x \mid N(t) = n\} \Pr\{N(t) = n\}$$

$$= \sum_{n=0}^{\infty} \Pr\{V(t) \le x,\ N(t) = n\}$$

If $n=0$, $P_r\{V(t)\le x, N(t)=0\} = P_r\{W_1 - t \le x\ ,\ W_1 > t\}$

$$= P_r\{W_1 \le t+x\ ,\ W_1 > t\} = F(t+x) - F(t)$$

If $n\ge 1$, $P_r\{V(t)\le x, N(t)=n\} = P_r\{W_{n+1} \le t+x\ ,\ W_n \le t < W_{n+1}\}$

$$= P_r\{W_n \le t\ ,\ t < W_{n+1} \le t+x\}$$

$$\cancel{= \int P_r\{t < T_{n+1} \le t+x \mid W_n = u}$$

$$= \int_0^t P_r\{t < u + T_{n+1} \le t+x \mid W_n = u\}\, dF_n(u)$$

$$= \int_0^t P_r\{t-u < T_{n+1} \le t+x-u\}\, dF_n(u)$$

$$= \int_0^t [F(t+x-u) - F(t-u)]\, dF_n(u)$$

Hence

$$P_r\{V(t)\le x\} = F(t+x) - F(t) + \sum_{n=1}^{\infty} \int_0^t [F(t+x-u) - F(t-u)]\, dF_n(u)$$

$$= F(t+x) - F(t) + \int_0^t [F(t+x-u) - F(t-u)] \sum dF_n(u)$$

$$= F(t+x) - F(t) + \int_0^t [F(t+x-u) - F(t-u)]\, dM(u) \quad \longleftarrow$$

If $F(t)$ has a density function $f(t)$,

$$m(t) = \frac{d}{dt} M(t) = \sum_{n=1}^{\infty} f_n(t)$$

$$M(t) = \int_0^t m(t)dt$$

Laplace & Laplace - Stieltjes Transforms

① Laplace : $\mathcal{L}\{G(t)\} \equiv \int_0^\infty G(t)e^{-st}dt$

② Laplace - Stieltjes : $\mathcal{LS}\{G(t)\} \equiv \int_0^\infty e^{-st}dG(t) \leftrightarrow \mathcal{L}\{\frac{dG}{dt}\}$

E.g. $G(t) = f(t) = $ p.d.f. $\mathcal{L}\{f(t)\} = E\{e^{-st}\}$

$G(t) = F(t) = $ d.f. , $\mathcal{LS}\{F(t)\} = E\{e^{-st}\}$

Example : Machines fail at rate λ

$$F(t) = 1 - e^{-\lambda t}$$

$$E\{N(t)\} = M(t) = \sum_{n=1}^{\infty} F_n(t)$$

$$m(t) = \sum f_n(t) = \sum_{n=1}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} = \lambda e^{-\lambda t} e^{\lambda t} = \lambda$$

$$M(t) = \int_0^t m(t)dt = \lambda t$$

Passage time in Markov process an easier route thru "renewal theory"? See 6.536 notes.

Present Age Distribution

$$U(t) = t - W_{N(t)}$$

Theorem: $\Pr\{U(t) \le x\} = \begin{cases} 1 & \text{if } t \le x \\ \int_{t-x}^{t} [1 - F(t-u)]\, dm(u) & \text{if } t > x \end{cases}$

Proof: Homework

Limit Theorems for Renewal Processes

Theorem: $\frac{m(t)}{t} \to \frac{1}{E\{T\}}$ as $t \to \infty$

i.e., in the limit as $t \to \infty$, arrival rate is uniform over the infinite horizon on an ensemble average basis

If $E\{T\} = +\infty$, then $\frac{m(t)}{t} \to 0$ as $t \to \infty$

Theorem: $\frac{N(t)}{t} \to \frac{1}{E\{T\}}$ as $t \to \infty$

This is a stronger statement; it says that every time the renewal process is run, the sample average of arrivals per unit time converges as $t \to \infty$.

Theorem: if $\mu \equiv E\{T\} < \infty$ then
$\sigma^2 = \mathrm{Var}\{T\} < \infty$

$$\frac{\mathrm{Var}\{N(t)\}}{t} \to \frac{\sigma^2}{\mu^3} \quad \text{as } t \to \infty$$

and $\Pr\left\{ \frac{N(t) - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \le y \right\} \to \mathrm{Normal}(0,1)$

Smith's
<u>Theorem</u>: Suppose T is a non-lattice r.v.
Let $Q(x)$ be a function $\ni$

$$\left.\begin{array}{l} Q(x \le 0) = 0 \\ Q(x) \ge 0 \\ \int_0^\infty Q(t)\,dt < \infty \\ Q(x) \text{ is non-increasing for } x \ge 0 \end{array}\right\} x \ge 0$$

Then
$$\lim_{t\to\infty} \int_0^t Q(t-s)\,dM(s) = \frac{1}{\mu}\int_0^\infty Q(s)\,ds$$

<u>Theorem</u>: If T is a non-lattice r.v., then for any $h > 0$

$$M(t+h) - M(t) \to \frac{h}{\mu} \quad \text{as } t \to \infty$$

This can be proved by the immediately preceeding theorem by using $Q(t) = 1$ for $0 < t < h$ & 0 otherwise.

Equilibrium Results for Renewal Processes

Assume a non-lattice r.v. & $\mu \equiv E\{T\} < \infty$

Find limiting distribution of $V(t) \equiv W_{N(t)+1} - t$:

Show: $P_r\{V(t) > x\} = 1 - F(t+x) + \int_0^t [1-F(t+x-u)]\,dM(u)$

$$P_r\{V(t) \le x\} \overset{?}{=} F(t+x) - \int_0^t [1-F(t+x-u)]\,dM(u)$$

$$= F(t+x) - M(t) + \int_0^t F(t+x-u)\,dM(u)$$

$$= F(t+x) - F(t) - F(t) * M(t) + \int_0^t \cdots$$

$$= F(t+x) - F(t) + \int_0^t [F(t+x-u) - F(t-u)]\,dM(u)$$

$$= P_r\{V(t) \le x\}$$ ✓ by thm on excess life distr

$$\lim_{t\to\infty} P_r\{V(t) > x\} = \lim \left\{1 - F(t+x) + \int_0^t [1-F(t+x-u)]\,dM(u)\right\}$$

but $\lim_{t\to\infty} [1-F(t+x)] = 0$ so

$$\lim_{t\to\infty} P_r\{V(t) > x\} = \lim \int_0^t [1-F(t+x-u)]\,dM(u)$$

let $Q(t-u) = \begin{cases} 1-F(t+x-u) & \text{if } 0 \le u < t \text{ or if } \underline{0 < t-u} \\ 0 & \text{otherwise} \end{cases}$

Since $Q(\cdot)$ is a distr fn, it satisfies the 3 condx of Smith's thm (page 11):

$$\lim P_r\{V(t) > x\} = \lim \int_0^t [1-F(t+x-u)]\,dM(u)$$

$$= \frac{1}{\mu}\int_0^\infty [1-F(y+x)]\,dy$$

$$= \frac{1}{\mu}\int_x^\infty [1-F(s)]\,ds$$

Or, $\lim_{t\to\infty} P_r\{V(t) \le x\} = 1 - \frac{1}{\mu}\int_x^\infty [1-F(s)]ds$

$$= \frac{1}{\mu}\left[\int_0^\infty [1-F(s)]ds - \int_x^\infty [1-F(s)]ds\right]$$

$$= \frac{1}{\mu}\int_0^x [1-F(s)]ds \equiv F_e(x) \longleftarrow$$

(Show in homework that $U(t)$ has the same limiting distribution; $U(t) = t - W_{N(t)}$.)

Hence, as $t\to\infty$ & we are in equilibrium, the probability that we must wait up to x seconds for the next arrival is the same as the probability that the last event was up to x seconds in the past.

Delayed Renewal Process

A delayed renewal process is defined to be a sequence of positive r.v.'s $T_1, T_2, \dots$ such that T_1 has the distribution fn H, while $T_2, T_3, \dots$ are i.i.d. with d.f. F.

A tilde over any symbol mean that it refers to a delayed renewal process; no tilde ⇒ a simple renewal proc. Then

$$\tilde{M}(t) \equiv E\{\tilde{N}(t)\}$$

(1) $\tilde{F}_n(x) = P_r\{\tilde{W}_n \le x\} = P_r\{T_1 + \cdots + T_n \le x\}$

$$= H(x) \quad \text{if } n=1$$

$$= H(x) \circledast \overset{*}{F}_{n-1}(x) = H(x) \circledast \overbrace{F(x) \circledast F(x) \cdots}^{n-1}$$

② As before,

$$\tilde{M}(t) = \sum_{n=1}^{\infty} \tilde{F}_n(t) = H(t) + \sum_{n=2}^{\infty} H * F_{n-1}(t)$$

$$= H(t) + H(t) \circledast \sum_{n=1}^{\infty} F_n(t)$$

$$= H(t) + H(t) \circledast M(t)$$

③ Suppose $H(x) = F_e(x)$. Then $H(x) = \frac{1}{\mu}\int_0^x [1-F(s)]\,ds$

$$= \frac{1}{\mu}\int_0^x [1-F(x-u)+F(x-u)-F(u)]\,du$$

$$= \frac{1}{\mu}\int_0^x [1-F(x-u)]\,du + \frac{1}{\mu}\int_0^x [F(x-u)-F(u)]\,du$$

$$= \frac{1}{\mu}\int_0^x [1-F(x-u)]\,du$$ could show directly

Let $I(t) = \begin{cases} 0, & t \le 0 \\ t, & t > 0 \end{cases}$

$$F_e(x) = \frac{1}{\mu}[1-F(x)] \circledast I(x)$$

Then $\tilde{m}(t) = F_e(t) + F_e(t) * M(t)$

$$= \frac{1}{\mu}\{(1-F)*I + (1-F)*I*M\}$$

$$= \frac{1}{\mu}\{(1-F)*I + I*(\underbrace{M - M*F}_{=F \text{ by renewal eqn}})\}$$

$$= \frac{1}{\mu}\{(1-F)*I + I*F\} = \frac{1}{\mu}\, 1 * I$$

$$= \frac{1}{\mu}\int_0^t dI(t) = \frac{t}{\mu}$$

or $\frac{\tilde{m}(t)}{t} = \frac{1}{\mu}$ = constant for all $t > 0$.

(Not just in lim $(t \to \infty)$ as for simple renewal process)

(4) $P_r\{\tilde{V}(t) \le x\}$ where $\tilde{V}(t) \equiv \tilde{W}_{\tilde{N}(t)+1} - t$

$$= H(t+x) - H(t) + \int_0^t [F(t+x-u) - F(t-u)]\, d\tilde{M}(u)$$

as with the simple renewal process

If $H = F_e$,

$$P_r\{\tilde{V}(t) \le x\} = F_e(t+x) - F_e(t) + \int_0^t [F(x+t-u) - F(t-u)]\, d\tilde{M}(u)$$

$$= \frac{1}{\mu}\int_t^{t+x} [1-F(u)]\,du + \int_0^t [1-F(t-u)]\,d\tilde{M}(u) - \int_0^t [1-F(x+t-u)]\,d\tilde{M}(u)$$

$$= \frac{1}{\mu}\int_t^{t+x} [1-F(u)]\,du + \frac{1}{\mu}\int_0^t [1-F(s)]\,ds - \frac{1}{\mu}\int_0^t [1-F(x+t-u)]\,du$$

$t+x-u = s$

$$= \frac{1}{\mu}\int_0^{t+x} [1-F(u)]\,du - \frac{1}{\mu}\int_x^{t+x} [1-F(s)]\,ds$$

$$= \frac{1}{\mu}\int_0^x [1-F(s)]\,ds \equiv F_e(x) \longleftarrow$$

Inventory Example:

$$\text{Ordering cost} = \begin{cases} 0 & \text{if } x = 0 \\ K + cx & \text{if } x > 0 \end{cases}, \quad K \ge 0$$

Penalty cost $= p$ per unit shortage

Storage cost $= h$ " " in storage

An (s, S) policy is optimal for this type of cost structure & we want to find optimal values for s + S.

Demands are a renewal process $\{T_i\}$ where T_i = demand in period i.

Let $\Delta = S - s$

Order when $T_1 + \cdots + T_K > \Delta$ if K is smallest such integer.

① Distribution of r.v. ρ = # periods between orders

Start with stock level S. Then by defn,

$$N(\Delta) = n \iff T_1 + \cdots + T_n \le \Delta < T_1 + \cdots + T_{n+1}$$

Then $\rho = n+1 \iff N(\Delta) = n$

$$P_r\{\rho = n+1\} = P_r\{N(\Delta) = n\} = F_{W_n}(\Delta) - F_{W_{n+1}}(\Delta)$$

Proof: $P_r\{N(\Delta) = n\} = P_r\{W_n \le \Delta,\ W_{n+1} > \Delta\}$
$= P_r\{W_n \le \Delta\} - P_r\{W_{n+1} \le \Delta\}$
$= F_{W_n}(\Delta) - F_{W_{n+1}}(\Delta)$

So

$$P_r\{N(\Delta) = n\} = P_r\{\rho = n+1\} = F_{W_n}(\Delta) - F_{W_{n+1}}(\Delta)$$

<u>Mean</u> is $\sum_{k=0}^{\infty} k\, P_r\{\rho = k\} = \sum_{k=1}^{\infty} k\, P_r\{N(\Delta) = k-1\}$

$$= \cancel{\sum_{k=1}^{\infty} k \left[F_{W_{k-1}}(\Delta) - F_{W_k}(\Delta)\right]}$$

$$= \sum_{n=0}^{\infty} (n+1)\, P_r\{N(\Delta) = n\} = \underline{\underline{M(\Delta) + 1 = E\{\rho\}}}$$

② Distribution of jump below Δ is the excess life distr:

$$P_r\{V(\Delta) \le x\} = F(\Delta + x) - F(x) + \int_0^{\Delta} F(\Delta + x - u) - F(\Delta - x)\,]\, dM(u)$$

1-

(3) Distr. of stock sizes at the beginning of a period:

$z_n \equiv$ stocks at end of period n
$=$ " " beginning " $n+1$ before order is placed & filled

$$z_{n+1} = \begin{cases} S - T_{n+1} & \text{if } z_n < s \\ z_n - T_{n+1} & z_n \geq s \end{cases} \qquad \begin{matrix} n = 0, 1, \ldots \\ -\infty < z_{n+1} < S \end{matrix}$$

Let $y_{n+1} = S - z_{n+1}$ so $0 < y_{n+1} < \infty$

Assume $f(t) = \frac{dF}{dt}$ & $g_{n+1}(t) =$ p.d.f. of y_{n+1}

$$g_{n+1}(t) = \left[\int_{S-s}^{\infty} g_n(u)\,du\right] f(t) + \int_0^{S-s} g_n(u)\, f(t-u)\,du$$

$\lim_{n\to\infty} g_{n+1} \to g$ a stationary pdf.

$$g(t) = f(t)\int_\Delta^\infty g(u)\,du + \int_0^\Delta g(u) f(t-u)\,du$$

$$0 < t \leq \Delta, \; g(t) = f(t)\int_\Delta^\infty g(u)\,du + \int_0^t g(u) f(t-u)\,du$$
$$= Af + g*f = Af + (Af + g*f)*f = Af + Af^{(2)} + g*f^{(2)}$$
$$= \cdots = A(f + f^{(2)} + f^{(3)} + \cdots) + O(n)$$
$$= A\,m(t)$$

$$t > \Delta, \; g(t) = Af(t) + \int_0^\Delta g(u)\, f(t-u)\,du$$
$$= Af(t) + \int_0^\Delta A\, m(u) f(t-u)\,du$$
$$= A\left\{ f(t) + \int_0^\Delta m(u) f(t-u)\,du \right\}$$

$$1 = \int_0^\Delta g(u)\,du + \int_\Delta^\infty g(u)\,du = A\int_0^\Delta m(u)\,du + A$$
$$= A[M(\Delta) + 1] \quad \text{so} \quad A = \frac{1}{1 + M(\Delta)}$$

Hence

$$g(t) = \begin{cases} \dfrac{m(t)}{1 + M(\Delta)} & 0 < t \leq \Delta \\ \dfrac{1}{1+M(\Delta)}\left[f(t) + \int_0^\Delta f(u)\, m(t-u)\,du\right] & t > \Delta \end{cases}$$

An interpretation of $g(t)$

$0 < t \leq \Delta$, $\quad P_r\{y \leq t\} = \dfrac{M(t)}{1+M(\Delta)} = \dfrac{M(t)}{M(\Delta)} \dfrac{M(\Delta)}{1+M(\Delta)}$

(1) $\dfrac{M(t)}{M(\Delta)} = P_r\{y \leq t / y \leq \Delta)$ where y is accumulated demand since last order

(2) $\dfrac{M(\Delta)}{1+M(\Delta)} = P_r\{y \leq \Delta\}$

$t > \Delta$: $\quad g(t) = \left[f(t) + \cancel{} \int_0^\Delta m(u) f(t-u)\, du\right] \dfrac{1}{1+M(\Delta)}$

excess-life density $\qquad P_r\{$

Inventory model continued

We have the distribution of $y = S - z$; we now want the distribution of $z = S - y$.

The density $z(x)$ for z is just $g(S-x)$, $-\infty < x < S$

$$z(x) = g(S-x) = \begin{cases} \dfrac{m(S-x)}{1+M(\Delta)}, & s \le x < S \\ \dfrac{1}{1+M(\Delta)}\left[f(S-x) + \int_0^\Delta m(u) f(S-x-u)\,du\right], & x < s \end{cases}$$

Example of above model

$f(t) = e^{-t}$, so $M(t) = t$ & $m(t) = 1$.

Excess life density:

$$f(S-x) + \int_0^\Delta f(S-x-u) m(u)\,du = e^{-(S-x)} + e^{-(S-x)} \int_0^\Delta e^{+u}\,du$$

$$= e^{-(S-x)} e^{\Delta} = e^{-(s-x)}$$

Hence: $$z(x) = \begin{cases} \dfrac{1}{1+\Delta} & s \le x < S \\ \dfrac{e^{-(s-x)}}{1+\Delta} & x < s \end{cases}$$

Define $L(x) = \begin{cases} K + c(S-x) + hS - p\min\{0,x\} & \text{if } x < s \\ hx & \text{if } s \le x < S \end{cases}$

Then total expected cost is (per period)

$$C(s,S) = \int_{-\infty}^{S} L(x) z(x)\,dx$$

$$= \int_{-\infty}^{0} z(x)\left[K + c(S-x) + hS - px\right] dx$$

$$+ \int_0^s z(x)\left[K + c(S-x) + hS\right] dx$$

$$+ \int_s^S z(x)\, hx\, dx$$

$$= K\,P_r\{z \le s\} + c\int_{-\infty}^{s}(s-x)\,z(x)\,dx + hS\,P_r\{z \le s\}$$

$$-p\int_{-\infty}^{0} x\,z(x)\,dx + \int_{0}^{S} hx\,z(x)\,dx$$

$$\cancel{= (K+h)\,P_r\{z \le s\}}$$

$$\cancel{= (K+h)\frac{1}{1+\Delta} + p\frac{e^{-s}}{1+\Delta} + c\frac{s-1}{1+\Delta} + h}$$

$$= \frac{K}{1+\Delta} + c\frac{S}{1+\Delta} - c\frac{s-1}{1+\Delta} + \frac{hS}{1+\Delta} + \frac{pe^{-s}}{1+\Delta} + \frac{h(S^2-s^2)}{2(1+\Delta)}$$

$$= \frac{1}{1+\Delta}\left[K + c(\Delta+1) + h\left[\frac{\Delta(2s+\Delta)}{2} + s + \Delta\right] + pe^{-s}\right]$$

$$= c + \frac{1}{1+\Delta}\left[K + pe^{-s} + h\left(\Delta s + \frac{\Delta^2}{2} + s + \Delta\right)\right]$$

$$\frac{\partial C}{\partial s} = \frac{1}{1+\Delta}\left[h\Delta + h - pe^{-s}\right] = 0 \Rightarrow \boxed{e^{-s} = \frac{h(1+\Delta)}{p}}$$

$\frac{\partial C}{\partial \Delta} = 0$ & $\frac{\partial C}{\partial s} = 0$ $\Rightarrow$ after algebra $\boxed{\begin{aligned}\Delta &= \sqrt{\tfrac{2K}{h}} \\ s &= -\ln \frac{h\left(1+\sqrt{\frac{2K}{h}}\right)}{p}\end{aligned}}$

Note if $K=0$, $\Delta = 0$ & order up to $s = S$ every period. $s = -\ln\frac{h}{p}$

This holds for $s \ge 0$ or $\frac{h}{p}\left(1+\sqrt{\frac{2K}{h}}\right) \ge 1$

If s comes out < 0, redefine limits in $C(s,S)$

Then:
$$\left.\begin{aligned}\Delta &= \frac{p}{h}e^{-s} - s - 1 \\ \Delta &= \sqrt{\frac{2K}{h} + s(s-2)}\end{aligned}\right\}$$

Lindley's integral eqn method for queuing: GI/G/1

X_n = waiting time in queue for n^{th} customer $\sim W_n(x)$
T_n = service time for n^{th} customer; mean $\frac{1}{\mu}$; $\sim F(t)$ i.i.d.
A_n = time between arrivals n and $n+1$; mean $\frac{1}{\lambda}$; $\sim G(t)$ i.i.d.
$D_n = T_n - A_n \sim K(t)$ i.i.d.

$$P_r\{D_n \le t\} = P_r\{T_n - A_n \le t\} = \int_0^\infty P_r\{T_n - A_n \le t \mid A_n = x\}\, dG(x)$$

$$= \int_0^\infty F(t+x)\, dG(x) = K(t) = \begin{cases} K_+(t) = \int_0^\infty F(t+x)\, dG(x) & t \ge 0 \\ K_-(t) = \int_{-t}^\infty F(t+x)\, dG(x) & t < 0 \end{cases}$$

$$x_{n+1} = \begin{cases} x_n + T_n - A_n & \text{if } > 0 \\ 0 & \text{if } \le 0 \end{cases}$$

$x_n + D_n > 0$

$$P_r\{x_{n+1} \le x\} = P_r\{x_n + D_n \le x\}$$

$$= W_{n+1}(x) = \int_0^\infty P_r\{x_n + D_n \le x \mid x_n = y\}\, dW_n(y)$$

$$= \int_0^\infty K(x-y)\, dW_n(y)$$

<u>Theorem</u>: Excluding D/D/1, $\exists$ a limiting equil distribn of W_n indep of $W_1 \Leftrightarrow E(D) < 0$; i.e., $\rho < 1$

$$\boxed{W(x) = \int_0^\infty K(x-y)\, dW(y) \qquad x \ge 0}$$ <u>Wiener-Hopf</u>

<u>Example</u>: deterministic arrivals at intervals of T.
exponential service time, mean $1/\mu$.

Then $G(t) = \begin{cases} 0 & t < T \\ 1 & t \ge T \end{cases}$

$$f(t) = \mu e^{-\mu t}, \quad t \geq 0$$

Then $K(t) = F(t+T)$ or $k(t) = f(t+T) = \mu e^{-T} e^{-\mu t}$, $t > -T$

& $W(x) = \int_0^\infty K(x-y)\,dW(y) = \int_{-\infty}^x W(x-y)\,dK(y)$

$$= \int_{-T}^x W(x-y)\,\mu e^{-(y+T)}\,dy \qquad x-y=z$$

$$= \int_0^{x+T} W(z)\,\mu e^{-\mu(T+x-z)}\,dz$$

Try $W(x) = 1-(1-\rho_0)e^{-\mu\rho_0 x}$

To get ρ_0, $W(0) = \int_0^T W(z)\,\mu e^{-\mu(T-z)}\,dz$

$$\rho_0 = \int_0^T \mu e^{-\mu(T-z)}\,dz - (1-\rho_0)\int_0^T \mu e^{-\mu T} e^{+\mu z} e^{-\mu\rho_0 z}\,dz$$

$$= 1-e^{-\mu T} - (1-\rho_0)e^{-\mu T}\int_0^T e^{+\mu(1-\rho_0)z}\,dz$$

$$= 1-e^{-\mu T} - \cancel{(1-\rho_0)}\, e^{-\mu T}\frac{1}{\cancel{1-\rho_0}}\left[e^{\mu(1-\rho_0)T}-1\right]$$

$$= 1-\cancel{e^{-\mu T}} - e^{-\mu\rho_0 T} + \cancel{e^{-\mu T}} = \underline{1-e^{-\mu\rho_0 T} = \rho_0}$$

$$\Rightarrow W(x) = 1-e^{-\mu\rho_0(T+x)}$$

Plug into integral eqn:

$$1-e^{-\mu\rho_0(T+x)} \overset{?}{=} \int_0^{x+T}\left[1-e^{-\mu\rho_0(T+z)}\right]\mu e^{-\mu(T+x-z)}\,dz$$

$$1-e^{-\mu\rho_0 y} \overset{?}{=} \int_0^y \mu e^{-\mu(y-z)}\,dz - \int_0^y \mu e^{-\mu(1+\rho_0)y} e^{-\mu z}\,dz$$

$$\overset{?}{=} e^{-\mu y}\left[e^{\mu y}-1\right] - e^{-\mu(1+\rho_0)y}\left[1-e^{-\mu y}\right]$$

$$\overset{?}{=} 1-e^{-\mu y} - e^{-\mu(1+\rho_0)y} + e^{-\mu(2+\rho_0)y}$$

$$\overset{?}{=} \int_0^{x+T}\mu e^{-\mu(T+x)} e^{\mu z}\,dz - \int_0^{x+T}\mu e^{-\mu(1+\rho_0)(T+x)} e^{+\mu(1-\rho_0)z}\,dz$$

$$\overset{?}{=} e^{-\mu(T+x)}\left[e^{\mu(T+x)}-1\right] - e^{-\mu(1+\rho_0)(T+x)}\left[e^{\mu(1-\rho_0)(x+T)}-1\right]\frac{1}{1-\rho_0}$$

$$W(x) = 1 - (1-P_0)e^{-\mu P_0 x} \qquad x \geq 0$$

is the dist of time in queue. ~~given a wait~~

Given a wait > 0, $W(x \mid x>0) = 1 - e^{-\mu P_0 x}$.

Theorem: If $F(t)$ is exponential, then $W(x|x>0)$ is exponential, irrespective of $G(t)$.

$E_n/E_m/1$

$$f(t) = \frac{(m\mu)^m}{(m-1)!} e^{-m\mu t} t^{m-1}$$

$$g(t) = \frac{(n\lambda)^n}{(n-1)!} e^{-n\lambda t} t^{n-1}$$

Assume soln: $w(x) = \sum_{r=0}^{N} \alpha_r e^{-\beta_r x}$, $\alpha_0 = 1$, $\beta_0 = 0$, $Re\{\alpha_r\} > 0$.

$M/D/1$

service time S

arrival rate λ

$$w(x) = (1-\lambda S) e^{\lambda x} \sum_{j=0}^{k} \frac{(j\lambda S - \lambda x)^j}{j!} e^{-j\lambda S}, \quad k = 0, 1, 2, \ldots \quad kS < x \leq (k+1)S$$

Markov Chains

Read Cox & Miller Chap 3 to p 118.

Definition: A Markov process is a stochastic process $\{X_n\}_{n=0}^{\infty}$ with state space $S = (0, 1, 2, \dots)$ or $(0, 1, \dots, S)$

where $P_r\{X_{n+1} = j \mid X_m = i_m, m = 0, 1, \dots, n\} = P_r\{X_{n+1} = j \mid X_n = i_n\}$

$$= P_{ij}^{n,\,n+1}$$

When $P_{ij}^{n,\,n+1} = P_{ij}$, the process is homogeneous.
We will deal only with homogeneous processes.

$$P = (P_{ij}) \quad \text{where } \sum_j P_{ij} = 1$$

$$P^n = \text{matrix of } n\text{-step transition probabilities} \quad \text{where } \sum_j P_{ij}^n = 1$$

p^0 = vector of initial probabilities $= (p_0^0, p_1^0, \dots)$

$$p^n = p^0 P^n$$

p_i^n = prob proc in state i at time n.

In equilibrium (if $\exists$), $p^n \to \pi$

[Note no problem if ∞ many states since
$$p_j^n = \sum_{i=1}^{\infty} p_i^0 P_{ij}^n \le \sum p_i^0 = 1]$$

Chapman-Kolmogorov Eqns

Let $P_{ij}(r,s) = P_r\{X_s = j \mid X_r = i\}$

Then for any $m, n, \& u \ni 0 \le m < u < n$

$$\boxed{P_{jk}(m,n) = \sum_i P_{ji}(m,u)\,P_{ik}(u,n)}$$

Not a sufficient condition for a Markov process.

Example: one-dimensional random walk.
State space $(0, 1, 2, \ldots)$

$$P = \begin{bmatrix} r_0 & p_0 & 0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots \\ 0 & 0 & & & & \\ \vdots & \vdots & & & & \end{bmatrix}$$

where $p_0, r_0 \ge 0 \;\&\; r_0 + p_0 = 1$
$p_i > 0,\ q_i > 0,\ r_i \ge 0 \;;\; \&\; p_i + q_i + r_i = 1$

[Might interpret as gambler v. ∞ wealth opponent]

Markov process as queuing model:
(1) A customer is served each period if present
(2)

ξ_n = # customers arriving in n^{th} period; i.i.d.
$P_r\{\xi_n = k\} = a_k$

State $\equiv$ # in line waiting at start of each period (before svc)

$$X_n = \begin{cases} X_{n-1} + \xi_{n-1} - 1 & \text{if } X_{n-1} \ge 1 \\ \xi_{n-1} & \text{if } X_{n-1} = 0 \end{cases} = \max\{X_{n-1} - 1, 0\} + \xi_{n-1}$$

If $E(\xi_n) = \sum_{k=0}^{\infty} k\, a_k$ >1, then queue blows up as $n\to\infty$.

<1, then queue should stabilize

$=1$, unstable (null-recurrent chain)

<u>Imbedded Markov chain for a queueing process</u> ~~(M/G/1)~~

X_n = # cust in system <u>just after</u> a service is completed.
for M/G/1

X_n = # cust in system just after an arrival
for GI/M/1.

<u>Classification of states of a Markov Chain</u>.

<u>Accessible</u>: j is accessible from i if

$P_{ij}^{n} > 0$ for some $n \geq 0$; then $i \to j$

If $i \to j$ & $j \to i$, then i & j <u>communicate</u>.

The $\leftrightarrow$ relation partitions the state space.

A Markov chain is <u>irreducible</u> if $\leftrightarrow$ relation reduces to only one class.

The <u>period</u> of state i, $d(i)$, is the greatest common divisor of all integers $n \geq 1$ for which $P_{ii}^{n} > 0$.

~~If any state~~

A chain of period 1 is <u>aperiodic</u>.

Theorem: If $i \Leftrightarrow j$, then $d(i) = d(j)$

Theorem: ~~If~~ $\exists$ an integer $N_i \ni p_{ii}^{nd(i)} > 0$ for $n \geq N_i$

Corollary: If $p_{ji}^m > 0$, then $p_{ji}^{m+nd(i)} > 0$ for all n large enough.

Recurrence:

$$f_{ii}^n \equiv P_r\{x_n = i,\ x_j \neq i,\ j = 1, 2, \ldots, n-1 \mid x_0 = i\}$$
$$= P_r\{\text{first return to state } i \text{ occurs at the } n^{th} \text{ transition}\}$$

$$f_{ii}^1 = p_{ii}$$

Theorem: $p_{ii}^n = \sum_{k=0}^{n} f_{ii}^k \, p_{ii}^{n-k}$ gives f_{ii} recursively.

e.g. $p_{ii}^2 = f_{ii}^1 p_{ii} + f_{ii}^2 \Rightarrow f_{ii}^2 = p_{ii}^2 - f_{ii}^1 p_{ii}$

Define $f_{ii}^0 \equiv 0$

Proof: Consider all possible realizations for which $x_0 = i$, $x_n = i$ and the first return to i is at k^{th} transition.

Similarly, define

$$f_{ij}^k \equiv P_r\{\text{first passage from } i \text{ to } j \text{ is at } k^{th} \text{ transition}\}$$

Then $p_{ij}^n = \sum_{k=0}^{n} f_{ij}^k \, p_{jj}^{n-k}$ with $f_{ij}^0 \equiv 0$.

Define the generating fns:

$$P_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}(z)$$

$$F_{ij}(z) = \sum_{n=0}^{\infty} f_{ij}(z)$$

Then $P_{ii}(z)F_{ii}(z) = \sum_{n=1}^{\infty} p_{ii}^{n} z^{n} = P_{ii}(z) - 1$

or $P_{ii}(z) = \frac{1}{1 - F_{ii}(z)}$

Similarly, $P_{ij}(z) = F_{ij}(z)P_{jj}(z)$

<u>Defn</u>: State i is <u>recurrent</u> iff $\sum_{n=1}^{\infty} f_{ii}^{n} = 1$. A non-recurrent state is called <u>transient</u>.

<u>Abel's lemma</u>:

(a) If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{\substack{z\to 1 \\ \text{from below}}} \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_k = a$.

(b) If $a_k \geq 0$ & $\lim_{\substack{z\to 1 \\ \text{from below}}} \sum_{k=0}^{\infty} a_k z^k = a \leq \infty$,

then $\sum_{k=0}^{\infty} a_k = \lim_{N\to\infty} \sum_{k=0}^{N} a_k = a$.

<u>Theorem</u>: A state is recurrent iff $\sum_{n=1}^{\infty} p_{ii}^{n} = \infty$.

<u>Proof</u>: If recurrent, $\sum_{n=1}^{\infty} f_{ii}^{n} = 1$. By (a),

$\lim_{z\to 1} F_{ii}(z) = 1$

Then $\lim_{z\to 1} P_{ii}(z) = \infty$. or $\sum p_{ii}^{n} = \infty$ by (b).

~~Conversely $\sum p_{ii}^{n} = \infty$~~

Conversely, suppose $\sum f_{ii}^{n} < 1$; then by (a)

$\lim_{z\to 1} F_{ii}(z) < 1 \Rightarrow$ by (b) $\sum_{n=0}^{\infty} p_{ii}^{n} < \infty$

Corollary: If $i \leftrightarrow j$ and i is recurrent, then j is recurrent.

Proof: Since $i \leftrightarrow j$, $\exists\ m, n \geq 1 \ni \begin{cases} P_{ij}^{n} > 0 \\ P_{ji}^{m} > 0 \end{cases}$

By Chapman-Kolmogorov eqn,

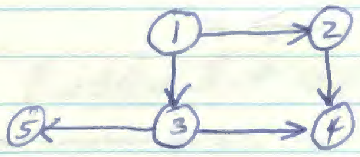
$$P_{jj}^{m+n+k} \geq P_{ji}^{m} P_{ii}^{k} P_{ij}^{n}$$

$$\& \quad \sum_{k=0}^{\infty} P_{jj}^{m+n+k} \geq P_{ji}^{m} P_{ij}^{n} \sum_{k=0}^{\infty} P_{ii}^{k}$$

So if $\sum_{k=0}^{\infty} P_{ii}^{k}$ diverges, so does $\sum_{k=0}^{\infty} P_{jj}^{m+n+k}$

So j is recurrent if i is.

Example: Finite state space Markov Chain w/5 equiv classes.



1, 2, 3 transient
4, 5 recurrent

$$P = \left[\begin{array}{c|c|c|c|c} Q_1 & R_{12} & R_{13} & 0 & 0 \\ 0 & Q_2 & 0 & R_{24} & 0 \\ 0 & 0 & Q_3 & R_{34} & R_{35} \\ 0 & 0 & 0 & Q_4 & 0 \\ 0 & 0 & 0 & 0 & Q_5 \end{array}\right]$$

Remarks:

(1) If one equiv class is accessible from a second, then the second ~~is not accessible from the first~~ is transient.

(2) In a finite state space Markov chain, there must be at least one recurrent class.

Example: One-dimensional random walk on + & − integers

$p = P_r\{\text{state increases one unit}\}$
$q = P_r\{\text{" decreases " "}\}$

$p + q = 1$

Hence, $P_{00}^{2n+1} = 0, \quad n \geq 0$

$$P_{00}^{2n} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{(n!)^2} p^n q^n$$

For large n (Stirling), $n! \approx n^{n+1/2} e^{-n} \sqrt{2\pi}$

$$P_{00}^{2n} \approx \frac{(pq)^n \, 2^{2n}}{\sqrt{\pi n}} = \frac{(4pq)^n}{\sqrt{\pi n}}$$

Then $\sum_{n=1}^{\infty} P_{00}^{n} \approx \sum_{n=1}^{\infty} \frac{(4pq)^n}{\sqrt{\pi n}}$

If $p = q = 1/2$, then $\sum_{n=1}^{\infty} P_{00}^{n} \approx \sum \frac{1}{\sqrt{\pi n}} \to \infty$

$\Rightarrow$ 0 is a recurrent state & since 0 communicates with all other states, the chain is a recurrent chain

If $pq = \rho < 1/4$, $\quad 4pq = \gamma < 1$

Then $\sum_{n=1}^{\infty} \frac{\gamma^n}{\sqrt{\pi n}} \leq \sum \gamma^n = \frac{1}{1-\gamma} < \infty$

$\Rightarrow$ the chain is transient & we drift off to $+\infty$ or $-\infty$.

The following theorem shows that a recurrent state will be visited infinitely often.

Defn: $Q_{ii} = \Pr\{\text{return to } i \ \infty\text{-often given start in state } i\}$

Theorem: State i is recurrent or transient $\Leftrightarrow Q_{ii} = 1$ or 0.

Proof:

Defn: $Q_{ii}^{N} \equiv \Pr\{\text{return to } i \text{ at least } N \text{ times} \mid \text{start in } i\}$

Then $Q_{ii}^{N} = \sum_{K=1}^{\infty} f_{ii}^{K} Q_{ii}^{N-1} = Q_{ii}^{N-1} f_{ii}^{*}$

Hence, $Q_{ii}^{N} = (f_{ii}^{*})^{N}$

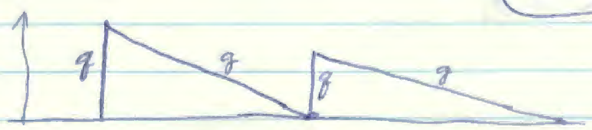
$$Q_{ii} = \lim_{N\to\infty} Q_{ii}^{N} = \begin{cases} 1 & \text{if } f_{ii}^{*} = 1 \quad (\text{recurrent}) \\ 0 & \text{if } f_{ii}^{*} < 1 \quad (\text{transient}) \end{cases}$$

$T = \frac{X}{g}$

$\int_0^{\infty} \phi(x)\, \psi\left(\frac{X}{T}\right) dx$?

Problem Set #4 ; #3

do this one



$q \sim \phi(q)$

$g \sim \psi(g)$

Find p.d.f. for stock level at random time.

View as renewal process with inter-order time distributed like q/g with p.d.f. $f(T)$

$$\Pr\{T \in (t, t+dt) \mid T > t\} = \frac{\Pr\{T \in (t, t+dt)\}}{\Pr\{T > t\}} = \frac{f(t)\,dt}{1 - F(t)}$$

Let $h(t) = \frac{f(t)}{1 - F(t)} \equiv$ hazard fn.

$$\int_0^t \frac{dF(u)}{1 - F(u)} = \int_0^t h(u)\,du$$

$$F(t) = 1 - e^{-\int_0^t h(u)\,du}$$

$$f(t) = h(t)\, e^{-\int_0^t h(u)\,du}$$

Now choose a random moment after equilibrium & ask what is the probability that the present age U is $\le t$.

$$P_r\{U \le t\} = \frac{1}{\mu}\int_0^t [1-F(u)]\,du$$

$$f_U(t) = \frac{1}{\mu}[1-F(t)]$$

Fix t: $z(i\,|\,t)$ = pdf of inventory level given t as present age.

$$z(i) = \frac{1}{\mu}\int_0^\infty z(i\,|\,t)[1-F(t)]\,dt$$

where $z(i\,|\,t) = P_r\{q - gt = i\}$

$$= \int_0^\infty P_r\{q-gt=i\,|\,g=s\}\,\psi(s)\,ds$$

$$= \int_0^\infty P_r\{q = st+i\}\,\psi(s)\,ds = \int_0^\infty \phi(st+i)\,\psi(s)\,ds$$

$$\phi(q) = \frac{1}{q_0}, \quad 0 < q < q_0$$

$$\psi(g) = \frac{1}{g_0}, \quad g_0 < g < 2g_0$$

$$h(t) = \frac{P_r\{q - gt = 0\}}{P_r\{q > gt\}}$$

$$P_r\{q - gt = 0\} = \int_0^\infty \phi(st)\,\psi(s)\,ds$$

3 cases

(1) $0 < t \le q_0/2g_0$, $\quad P_r = \int_{g_0}^{2g_0} \frac{ds}{q_0 g_0} = \frac{1}{q_0}$,

(2) $q_0/2g_0 < t \le q_0/g_0$ $\quad P_r = \int_{g_0}^{q_0/t} \frac{1}{q_0 g_0}\,ds = \frac{1}{q_0 g_0}\left[\frac{q_0}{t} - g_0\right]$

(3) $t > q_0/g_0$ $\quad P_r = 0$

Basic Limit Theorem for Markov Chains

For a recurrent, irreducible, aperiodic Markov chain,

(a) $\lim_{n\to\infty} P_{ii}^n = \left[\sum_{n=0}^{\infty} n f_{ii}^n\right]^{-1}$

(b) $\lim_{n\to\infty} P_{ji}^n = \lim_{n\to\infty} P_{ii}^n$

Theorem: In a positive recurrent aperiodic class with states $0, 1, 2, \ldots$

$$\lim_{n\to\infty} P_{jj}^n = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$$

$$\sum_{j=0}^{\infty} \pi_j = 1$$

$\{\pi_j\}$ is unique, given by $\begin{cases} \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \\ \sum \pi_j = 1 \end{cases}$

Proof:

$$1 = \sum_{j=0}^{\infty} P_{ij}^n \geq \sum_{j=0}^{M} P_{ij}^n$$

By basic limit theorem,

$$\lim_{n\to\infty} \sum_{j=0}^{M} P_{ij}^n = \sum_{j=0}^{M} \pi_j \leq 1$$

So $\sum_{j=0}^{\infty} \pi_j \leq 1$

By Chapman-Kolmogorov:

$$P_{ij}^{n+1} \geq \sum_{k=0}^{M} P_{ik}^n P_{kj}$$

Letting $n \to \infty$,

$$\pi_j \geq \sum_{k=0}^{M} \pi_k P_{kj}$$

$$\text{or } \pi_j \geq \sum_{k=0}^{\infty} \pi_k P_{kj}$$

Multiply by P_{ji} & sum on j

$$\pi_i \geq \sum_{j=0}^{\infty} P_{ji} \pi_j \geq \sum_j \sum_k \pi_k P_{kj} P_{ji} = \sum_k \pi_k \sum_j P_{kj} P_{ji}$$

$$= \sum_k \pi_k P_{ki}^2$$

Repeat to show $\pi_j \geq \sum_k \pi_k P_{kj}^n$ for all n.

Now suppose $\exists\ j, n \ni \pi_j > \sum_k \pi_k P_{kj}^n$

Then $\sum_j \pi_j > \sum_j \sum_k \pi_k P_{kj}^n = \sum_k \pi_k \sum_j P_{kj}^n = \sum_k \pi_k$

but this is a contradiction, so $\underline{\pi_j = \sum_k \pi_k P_{kj}^n \text{ all } n}$.

As $n \to \infty$, since $\sum \pi_k \leq 1$ and $P_{kj}^n \leq 1$,

$$\pi_j = \sum_{k=0}^{\infty} \pi_k \lim_{n\to\infty} P_{kj}^n = \sum_k \pi_k \pi_j$$

Hence $\underline{\sum \pi_k = 1}$ since $\pi_j > 0$ by recurrence assumption.

This shows $\exists$ stationary distribution; now need to show it is unique. Proof by contradiction:

Suppose $x = \{x_n\}$ satisfies $x_n \geq 0$, $\sum x_n = 1$, $x = xP$

Then $x_k = \sum_j x_j P_{jk} = \sum_j x_j P_{jk}^n$ as before

$n \to \infty$, $x_k = \sum x_j \lim_{n\to\infty} P_{jk}^n = \pi_k \sum x_j = \pi_k$.

Example: null recurrent chain: random walk with transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ q_1 & 0 & p_1 & 0 & \cdots \\ 0 & q_2 & 0 & p_2 & 0 & \cdots \\ \vdots \end{bmatrix}$$

Interested if $\exists$ a stationary distribution $\{\pi_k\}$; i.e., does $x = xP$ have a solution?

$$x_i = \sum_j x_j P_{ji} = p_{i-1} x_{i-1} + q_{i+1} x_{i+1}$$

where $p_{-1} = 0$ & $p_0 = 1$.

(1) $x_0 = q_1 x_1$

$$x_1 = x_0 + x_2 q_2 \quad = \frac{1}{q_1} x_0$$

$$x_2 = x_1 p_1 + x_3 q_3 \quad = \frac{1}{q_2}(x_1 - x_0) = \frac{p_1}{q_2 q_1} x_0$$

$$\vdots$$

$$x_3 = x_2 p_2 + x_4 q_4 \quad = \frac{p_1 p_2}{q_3 q_2 q_1} x_0$$

$$\vdots$$

By induction, $\underline{x_n = \frac{p_1 p_2 \cdots p_{n-1}}{q_1 q_2 \cdots q_n} x_0 = x_0 \prod_{k=0}^{n-1} \frac{p_k}{q_{k+1}}}$ w/ $p_0 = 1$

Must have $1 = x_0 + \sum_{n=1}^{\infty} x_0 \prod_{k=0}^{n-1} \frac{p_k}{q_{k+1}}$

$$x_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} \frac{p_k}{q_{k+1}}}$$

$$x_0 > 0 \iff \sum \prod \frac{p_k}{q_{k+1}} < \infty$$

E.g., in particular, if ~~[illegible]~~ $p_k = p$, $q_k = q$ for $k \geq 1$,

$\sum \prod \frac{p_k}{q_k} = \sum \left(\frac{p}{q}\right)^n \frac{1}{p}$ converges when $p < q$.

Null recurrent chain has $p > q$; states communicate, but "tendency" $\to \infty$

Example $P = \begin{bmatrix} 1/2 & 1/2 & & & \\ 1/2 & 1/2 & & & \\ 0 & 1/2 & 0 & 1/2 & 0 \\ & & & 1/2 & 1/2 \\ & & & 1/2 & 1/2 \end{bmatrix}$

$x = xP$ gives
$$\begin{aligned} x_1 &= 1/2(x_1+x_2) = x_2 \\ x_2 &= 1/2(x_1+x_2) = x_1 \\ x_3 &= 0 \\ x_4 &= 1/2(x_4+x_5) = x_5 \\ x_5 &= 1/2(x_4+x_5) = x_4 \end{aligned}$$

$1 = 2x_1 + 2x_4$ not unique

Absorbtion Probabilities

In HW #7, we prove: $P_{ij}^n \to 0$ as $n \to \infty$ if j is transient & $P_{ij}^n \to \pi_j$ as $n \to \infty$ for i, j in same aperiodic recurrent class.

What is limiting behavior of P_{ij}^n if i is transient & j is recurrent.

Consider $x_i^1 = \sum_{j \in T} P_{ij} \le 1, \quad i \in T$

& define recursively:

$$x_i^n = \sum_{j \in T} P_{ij} x_j^{n-1}.$$

x_i^n is the probability that, starting from i, the state of the process stays within T for the next n transitions. $x_i^n \le 1$ for all $n \ge 1$ because they are probabilities. Prove by induction that x_i^n is non-decreasing fn of n.

We have $x_i^2 = \sum_{j\in T} P_{ij} x_j^1 \le \sum_{j\in T} P_{ij} = x_i^1$

Suppose $x_j^n \le x_j^{n-1}$ for all $j \in T$

Then $0 \le x_i^{n+1} = \sum_{j\in T} P_{ij} x_j^n \le \sum_{j\in T} P_{ij} x_j^{n-1}$

Hence x_i^n is monotone non-increasing for $i \in T$ of n and $x_i^n \ge 0$, so $\lim_{n\to\infty} x_i^n = x_i$ exists.

x_i is probability of never being absorbed into a recurrent class starting from state $i \in T$, and it must be $x_i = 0$ for $i \in T$.

$$x_i = \lim_{n\to\infty} x_i^n = \lim_{n\to\infty} \sum_{j\in T} P_{ij} x_j^{n-1} = \sum_{j\in T} P_{ij} x_j \qquad , i \in T$$

If $Q \equiv (q_{ij}) = (P_{ij} \mid i,j \in T)$

~~$x = xQ$~~ $\quad Qx = x$ or $(I-Q)x = 0$.

Since $x=0$ is only soln (intuitive physical grounds), this $\Rightarrow$

$$(I-Q)^{-1} \text{ exists.}$$

Remarks:

If there are only a finite number of states, M, then $\exists$ no null recurrent states & not all states can be transient. To see this, let C = recurrent class & let $i \in C$

$$\sum_{j=0}^{M-1} P_{ij}^n = 1 \text{ for all } n \;\&\; P_{ij}^n = 0,\ j \notin C.$$

$$\Rightarrow \sum_{j=0}^{M-1} \pi_j = 1 \Rightarrow \exists \text{ a } k \in C \ni \pi_k > 0 \Rightarrow \text{class is positive recurrent.}$$

To see that all states cannot be transient in a finite chain, note $\sum_{j=0}^{M-1} P_{ij}^n = 1$ & obtain obvious contradiction if assume $\lim_{n\to\infty} P_{ij}^n = 0$ for <u>all</u> j.

Let $C_1, C_2, \ldots$ denote the recurrent classes.
$\pi_i(c)$ = Prob that process is ultimately absorbed into C if initial state is transient state i.

$\pi_i^n(c)$ = Prob that absorption occurs ~~for the first time is~~ at the n^{th} transition given initial transient state i.

$$\pi_i(c) = \sum_{n=1}^{\infty} \pi_i^n(c) \quad \leq 1$$

$$\pi_i^1(c) = \sum_{j\in C} P_{ij}$$

$$\pi_i^n(c) = \sum_{j\in T} P_{ij}\, \pi_j^{n-1}(c) \quad , n \geq 2$$

$$\pi_i(c) = \sum_{n=1}^{\infty} \pi_i^n(c) = \pi_i^1 + \sum_{n=2}^{\infty} \pi_i^n(c) \cancel{= \pi_i(c) + \sum_{j\in}}$$

$$= \pi_i^1(c) + \sum_{n=2}^{\infty} \sum_{j\in T} P_{ij}\, \pi_j^{n-1}(c)$$

$$= \pi_i^1(c) + \sum_{j\in T} P_{ij} \sum_{n=1}^{\infty} \pi_j^n(c)$$

$$= \pi_i^1(c) + \sum_{j\in T} P_{ij}\, \pi_j(c)$$

Let $\pi(c) = \begin{bmatrix} \pi_1(c) \\ \vdots \\ \pi_j(c) \\ \vdots \end{bmatrix}$ Then $\pi(c) = \pi^1(c) + Q\,\pi(c)$

$$(I-Q)\,\pi(c) = \pi^1(c)$$

$$\pi(c) = (I-Q)^{-1}\pi^1(c) \Rightarrow \text{unique soln for } \pi(c)$$

Remark: Either $\pi_i^1(c) > 0$ for some $i \in T$ or $\pi_i(c) = 0$ for all $i \in T$ & hence $\pi_i^n(c) = 0$ for all n; i.e. either we can be absorbed in C in one step from at least one step from T or C can never be reached.

Theorem: If $j \in C$ & C is aperiodic recurrent, then
$$\lim_{n\to\infty} P_{ij}^n = \pi_i(C) \lim_{n\to\infty} P_{jj}^n = \pi_i(C)\pi_j \quad \text{for } i \in T.$$

Proof in Karlin p. 71–72

Consider a finite chain with 2 aper. recur. classes & one aper. transient class.

$$P = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ S_1 & S_2 & S \end{bmatrix} ; \quad P^2 = \begin{bmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ S_1R_1 + SS_1 & S_2R_2 + SS_2 & S^2 \end{bmatrix}$$

$$P^n = \begin{bmatrix} R_1^n & 0 & 0 \\ 0 & R_2^n & 0 \\ A_n & B_n & S^n \end{bmatrix} \qquad A_n = S_1(R_1^{n-1} + SR_1^{n-2} + \cdots + S^{n-2}R_1 \cancel{+} S^{n-1})$$

$$\lim_{n\to\infty} P^n = \begin{bmatrix} R_1^* & 0 & 0 \\ 0 & R_2^* & 0 \\ A^* & B^* & 0 \end{bmatrix} = P^*; \quad A^* = \begin{pmatrix} \pi_{k_2+1}(C_1)\pi_1 & \cdots & \pi_{k_2+1}(C_1)\pi_{k_1} \\ \vdots & & \vdots \\ \pi_{k_3}(C_1)\pi_1 & \cdots & \pi_{k_3}(C_1)\pi_{k_1} \end{pmatrix}$$

$$p_n = p_0 P^n \;; \text{ what is } \lim_{n\to\infty} p_n = p_0 P^* \;?$$

$\cancel{\lim_{n\to\infty} p_n = (\pi_1, \ldots, \pi_{k_1}, 0) \text{ if } p_0 = (1, 0, 0, \ldots, 0)}$

$$\lim_{n\to\infty} p_n = (\pi_1, \ldots, \pi_{k_1}, 0, \ldots, 0) \text{ if } p_0 = (1, 0, \ldots, 0)$$

Suppose: ~~$p_{0,1}$ $p_0, p_1 > 0$~~ ~~$p_0, p_{k_1+1} > 0$ & p_0~~ $\quad p_{0,1} > 0$, $p_{0,k_1+1} > 0$, $\Sigma = 1$

(i.e., randomly starting in 1st state of C_1 or C_2):

$$p_0 P^* = (p_{0,1}\pi_1, \ldots, p_{0,1}\pi_{k_1}; p_{0,k_1+1}\pi_{k_1+1}, \ldots, p_{0,k_1+1}; 0 \ldots)$$

<u>Absorbtion probabilities contd:</u>

Partition a Markov chain into equivalence classes by the relation $i \Leftrightarrow j$.

<u>Defn</u>: A class C is closed if $\exists$ no states outside the class which are accessible from states within C.

Any recurrent class R is closed; if a transient class T is closed, then it must have an ∞ # of states

~~Ch~~

An arbitrary non-closed transient class T has ∞ # of states, it is easy to show that $x_i = 0$, $i \in T$ where x_i is the probability of never leaving T starting in i.

$\therefore$ If T has ∞ # of states $\{1, 2, \ldots\}$ & we collapse all states not in T into one state $i = 0$,

$$\tilde{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \tilde{P}_{10} & P_{11} & P_{12} & \cdots \\ \tilde{P}_{20} & & & \\ \vdots & & & \end{bmatrix}$$ where $\tilde{P}_{i0} = \sum_{j \notin T} P_{ij}$ & P is original matrix for T

In homework, we proved that $\lim_{n\to\infty} P_{ij} = 0$ if $j \in T$

$$\therefore \tilde{P}^n \to \tilde{P}_n^* = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \\ & \vdots & & & \end{bmatrix}$$

Imbedded Markov Chains

M/G/1 queueing model

Define $a_r = P_r\{N=r\} = \int_0^\infty \frac{(\lambda t)^r}{r!} e^{-\lambda t} dF(t)$

service time distr.
arrival rate $= \lambda$.

Equilibrium distribution: $\pi = (\pi_0, \pi_1, \ldots)$

$\pi_i > 0$, $\sum \pi_i = 1$

$\pi = \pi P$ or $\pi_i = \sum_r \pi_r p_{ri}$

→ $\pi_i \equiv$ equilibrium probability that there are i customers in the system just after a service has been completed.

Assume π exists; $\pi_i = \pi_0 a_i + \sum_{r=1}^{i+1} \pi_r a_{i-r+1}$

Define $\pi(z) = \sum_{i=0}^\infty \pi_i z^i$

$A(z) = \sum_{i=0}^\infty a_i z^i$

Z-transforming above eqn gives

$$\pi(z) = \pi_0 A(z) + \frac{1}{z}[A(z)\pi(z) - \pi_0 a_0] - \frac{\pi_0}{z}[A(z) - a_0]$$

$$= \frac{\pi_0 A(z)\ (z-1)}{z - A(z)}$$

Note: $A(z) = F^*[\lambda(1-z)]$ where $F^*(s) = \int_0^\infty e^{-st} dF(t)$
Laplace-Stieltjes

To get π_0, note $A(1) = F^*(0) = 1 = \sum a_i$

~~$\pi(z) = \pi_0 A(z)$~~

$$A'(1) = \sum r a_r = E\{N\} = \lambda E\{T\} = \frac{\lambda}{\mu} \equiv \rho$$

Then $\pi(1) = \dfrac{\pi_0 A(1)}{1 - A'(1)}$ via L'Hospital & algebra

$$1 = \frac{\pi_0}{1-\rho} \text{ so } \underline{\pi_0 = 1-\rho}$$

Hence,
$$\pi(z) = \frac{(1-\rho)(z-1)\,A(z)}{z - A(z)} = \frac{(1-\rho)F^*[\lambda(1-z)](z-1)}{z - F^*[\lambda(1-z)]}$$

Direct derivation of expected line length for a customer just arriving under the condition that the system is in equilibrium.

$q \equiv$ # in line after a departure

$q' =$ # after the next departure

$$q' = q - 1 + \delta + N$$

N: arrivals during service period

δ: 1 or 0 as $q = 0$ or $q > 0$.

In equilibrium, $E\{q'\} = E\{q\}$ so $0 = -1 + E\{\delta\} + E\{N\}$

$$E\{\delta\} = 1 - E\{N\} = 1 - \rho$$

Squaring gives $(q')^2 = q^2 + 2q(-1 + N + \delta) + (N-1)^2 + 2(N-1)\delta + \delta^2$

$$E\{q'^2\} = E\{q^2\} + 2E\{q(-1+N+\delta)\} + E\{(N-1)^2 + 2(N-1)\delta + \delta^2\}$$

But $E\{q'^2\} = E\{q^2\}$, $q\delta = 0$, $\delta^2 = \delta$

so $0 = 2E\{Nq - q\} + E\{(N-1)^2\} + 2E\{\delta(N-1)\} + E\{\delta\}$

$$2E\{q(1-N)\} = E\{(N-1)^2\} + E\{(2N-1)\delta\}$$

$\because$ N & q are indep, so

$$2E\{q\}E\{1-N\} = E\{(N-1)^2\} + \overset{1-\rho}{E\{\delta\}}\,E\{2N-1\}$$

$$= E\{N^2\} - \rho E\{2N-1\}$$

$$\text{so } E\{q\} = \frac{E\{N^2\} - \rho(2\rho-1)}{2(1-\rho)} = \rho + \frac{1}{2}\,\frac{E\{N^2\}-\rho}{1-\rho}$$

Letting σ_s^2 = var of svc time

$$\text{var}(N) = \lambda^2\sigma_s^2$$

$$\text{so } E\{N^2\} = \lambda^2\sigma_s^2 + \rho^2$$

$$\text{so } E\{q\} = \rho + \frac{1}{2}\,\frac{\lambda^2\sigma_s^2+\rho^2}{1-\rho}$$

Capacity expansion & Probabilistic growth Model

Deterministic model with backlogs in demand:

Unit time $\equiv \Rightarrow$ demand is one unit per unit time

Installation costs that result from increase in capacity of X is:

$$i(x) = kx^a$$

Discount rate r

$C(x) \equiv$ sum of future discounted costs to ∞, starting at a regeneration point (provide x unit of excess capacity each time

$$C(x) = kx^a + e^{-rx}C(x)$$

$$= \frac{kx^a}{1-e^{-rx}}$$

$C(x)$ is convex, so min by differentiating, giving

$$a = \frac{rx}{e^{rx}-1}$$

Probabilistic (Bachelier – Wiener diffusion process) model:

Consider a process which changes every Δt units of time. With probability p, the discrete change in demand is ΔD & with prob $1-p = q$ there is a decrease of ΔD in demand.

Markov process: $$\begin{cases} D(t) = D(t-\Delta t) + \epsilon(t) \\ \epsilon(t) = \begin{cases} \Delta D & \text{w/ prob } p \\ -\Delta D & \text{w/ prob } q \end{cases} \end{cases}$$

As $\Delta D, \Delta t \to 0$, all times & ~~sup~~ demands become possible. In time t, mean & variance of demand are about

$$t(p-q)\frac{\Delta D}{\Delta t} \quad \& \quad 4pqt\,\frac{\Delta D^2}{\Delta t}$$

These must remain finite, so

$$\frac{\Delta D^2}{\Delta t} \;\&\; \frac{(p-q)\Delta D}{\Delta t} \text{ must remain finite}$$

Putting $\sigma^2 = \frac{\Delta D^2}{\Delta t}$, set $p = \frac{1}{2} + \mu\frac{\Delta D}{2\sigma^2}$

$q = \frac{1}{2} - \mu\frac{\Delta D}{2\sigma^2}$

The constants σ^2 & μ are the diffusion coefficient & the drift, respectively.

If $\mu = 0$, the process (random walk) is symmetric. In the limit $\Delta D, \Delta t \to 0$, $p = q = 1/2$ to avoid ∞ displacements.

The total displacement at time $t \approx n\Delta t$ is distributed like the sum of n Bernoulli trials:

$V_{D,n} \equiv \Pr\{\text{demand has grown by } D \text{ units after } n \text{ trials}\}$

$$v(D,t) \equiv \lim_{\substack{n\to\infty \\ \Delta t\to 0 \\ n\Delta t \to t}} V_{D,n}$$

$v(D,t)$ is Normal w/mean $\frac{t(p-q)\Delta D}{\Delta t} = t\,\frac{\mu}{\sigma^2}\frac{\Delta D^2}{\Delta t} = \mu t$

variance: $\sigma^2 t$

So $D(t)$ is Normal $(\mu t, \sigma^2 t)$

But we need first passage to x:

Let $u(t,x)$ = probability density that system is in state x when t units of time elapse ~~between steps a demand increase of x~~

$$C(x) = Kx^a + \int_0^\infty u(t,x) e^{-rt} C(x)\,dt$$

To get $u(t,x)$, note

$$u(t+\Delta t, x) = p\,u(t; x-\Delta x) + q\,u(t, x+\Delta x)$$

Taylor series expansion:

$$u(t, x-\Delta x) = u(t,x) - \Delta x \frac{\partial u}{\partial x} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + \cdots$$

$$u(t+\Delta t, x) = u(t,x) + (q-p)\Delta x \frac{\partial u}{\partial x} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + \cdots$$

$$= u(t,x) + \Delta t \frac{\partial u}{\partial t}$$

$$\Rightarrow \frac{\partial u}{\partial t} = (q-p)\frac{\Delta x}{\Delta t}\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\Delta x^2}{\Delta t}\frac{\partial^2 u}{\partial x^2}$$

$$\cancel{\frac{\partial u}{\partial t} = -\frac{\mu \Delta p}{\sigma^2}\frac{\partial u}{\partial x} +}$$

$$\frac{\partial u}{\partial t} = -\mu \frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}$$

Fokker-Planck equation

Feller shows that $v(t,x)$ also satisfies this eqn.

We want $v(t,x)$ for

$$C(x) = kx^a + \int_0^\infty v(t,x)e^{-rt}C(x)\,dt$$
$$= kx^a + C(x)\,\bar{v}(r,x)$$

Transforming both sides of the FP eqn gives

$$r\bar{v}(r,x) = -\mu \frac{d}{dx}\bar{v}(r,x) + \frac{\sigma^2}{2}\frac{d^2}{dx^2}\bar{v}(r,x)$$

This is a 2d order linear diff eqn w/ soln:

$$\bar{v}(r,x) = A(r)e^{\lambda_1 x} + B(r)e^{\lambda_2 x}$$

$$\lambda_1 = \frac{\mu}{\sigma^2}\left[1 + \sqrt{1 + \frac{2r\sigma^2}{\mu^2}}\right] \qquad \lambda_1 > 0$$

$$\lambda_2 < 0$$

$$\lambda_2 = -\frac{\mu}{\sigma^2}\left[1 - \sqrt{1 + \frac{2r\sigma^2}{\mu^2}}\right]$$

Boundary conditions:

$0 \le \bar{v}(r,x) \le 1 \quad \Rightarrow A(r) = 0$ since $\lambda_1 > 0$

$\bar{v}(r,0) = 1 \quad \Rightarrow B(r) = 1$

Hence,

$$\bar{v}(r,x) = e^{-\lambda_2 x}$$

$$C(x) = kx^a + e^{-\lambda_2 x}C(x)$$ [cf. determ case $\lambda_2 \leftrightarrow r$] : adjusted interest rate

$$C(x) = \frac{kx^a}{1 - e^{-\lambda_2 x}}$$

As in determ case, min C is at $x^* \ni a = \dfrac{\lambda_2 x^*}{e^{-\lambda_2 x^*} - 1}$

As σ^2 increases,

(1) $C(x^*)$ increases

(2) x^* increases

As $\sigma^2 \to 0$, $\lambda_2 \to r/\mu$

e.g. $\mu = 1$, $a = .5$, $r = .15$

σ^2	λ_2	x^*	$C(x^*)$
0	+.15	8.4	4.0
1	.14	9.0	4.2
4	.12	10.4	4.5
16	.09	14.3	5.3

~~Learning~~

8 16 1
10 9 20 16)126
7 5 11.2
4 1 14
8 8
2 8
13 4

theary

Learning theory:

Lights E_0, E_1 are turned on in some pattern. Subject is asked to guess in advance of each event which light will be turned on next; his guesses are A_0 or A_1.

How will subjects' guesses change in the long run for a given behavior by the experimenter. One such pattern: Choose action with fixed probability depending on guess:

$$\begin{array}{cc} & \begin{array}{cc} E_0 & E_1 \end{array} \\ \begin{array}{c} A_0 \\ A_1 \end{array} & \begin{bmatrix} 1-v & v \\ w & 1-w \end{bmatrix} \end{array}$$

If $v=0$, $w>0$, then A_0 is always reinforced but A_1 is only sometimes reinforced.

If $v+w=1$, then $1-v=w$ and $1-w=v$, and the probability of E_0 ~~or~~ E_1 is independent of guess.

Model of subject behavior: Subject acts as though he had stimulus elements (SR's), some connected to A_0 & some to A_1. He samples each SR w/ probability t. If K sampled SR are connected to A_0 & l to A_1, then subject guesses A_0 with probability $K/K+l$.

Represent the model as $s+1$ state Markov chain. State S_i occurs when i ~~do~~ SR are connected to A_1, $i=0,\dots,s$

(a) Given present state, what is the prob of A_i?
(b) How does this probability evolve?

$\cancel{\frac{(s-i)!\; i!}{k!(s-i-k)!\,(m-k)!\,(i-m+k)!}}$ ~~0011 0~~

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(a) $P_i(A_1) = P_r\{A_1 \mid s_i\}$

If sample size m is drawn from the s SR's; k are connected to A_0 & $l = m-k$ to A_1.

$$P_r\{A_1 \mid s_i, k, l\} = \binom{s-i}{k}\binom{i}{l} t^m (1-t)^{s-m} \frac{l}{m}$$

$$P_r\{A_1 \mid s_i\} = \sum_{k=0}^{s-i}\sum_{l=0}^{i} \binom{s-i}{k}\binom{i}{l} t^m (1-t)^{s-m} \frac{l}{m} + \underbrace{(1-t)^s \frac{i}{s}}_{\text{Prob}\{A_1 \mid \text{no sample}\}}$$

$$= \sum_{m=1}^{s} t^m (1-t)^{s-m} \sum_{k+l=m\geq 1} \binom{s-i}{k}\binom{i}{l}\frac{l}{m} + (1-t)^s \frac{i}{s}$$

$$= \sum_{m=1}^{s} \binom{s}{m}\frac{i}{s}\, t^m (1-t)^{s-m} + (1-t)^s \frac{i}{s}$$

$$= \sum_{m=0}^{s} \binom{s}{m}\frac{i}{s}\, t^m (1-t)^{s-m} = \frac{i}{s}$$

(b) Let $\gamma = \begin{bmatrix} 0 \\ 1/s \\ 2/s \\ \vdots \\ 1 \end{bmatrix}$

$$p^n = P^n \gamma \equiv \begin{pmatrix} p_0^n \\ \vdots \\ p_s^n \end{pmatrix}$$ where p_i^n = Prob of A_1 at the n^{th} trial given that before the first trial there were i SR's on A_1.

$$p_i^n = p_{i0}^n \cdot 0 + p_{i1}^n \frac{1}{s} + p_{i2}^n \frac{2}{s} + \cdots + p_{is}^n \cdot 1$$

p_0 is unknown.

i = # SR connected to A_1

If A_0E_0 or A_1E_0, i decreases (strictly for A_1E_0)
If A_0E_1 or A_1E_1, i increases

The combination of A_1 & E_0 with a transition from s_i down to s_j has probabilities vX where

$$X = (x_{ij}) \ \& \ x_{ij} = \begin{cases} \sum_{k=0}^{s-i} \binom{s-i}{k}\binom{i}{i-j} t^{i-j+k}(1-t)^{s-i+j-k} \frac{i-j}{i-j+k} & \text{if } j<i \\ 0 & \text{if } j \geq i \end{cases}$$

(To go from i down to j, need sample of size at least $k+i-j$, $k=0,\ldots,s-i$)

The downward transition from s_i to s_j by means of $A_0 E_0$ has prob $(1-v)(Y-X)$ where $Y = (y_{ij})$

$$y_{ij} = \begin{cases} \binom{i}{i-j} t^{i-j}(1-t)^{j} & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}$$

~~for $i>j$~~

for $i>j$, note:

$$\underbrace{\Pr\{\text{sample } i\text{-}j \text{ SR on } A_1 \text{ + guess } A_1\}}_{= x_{ij}} + \Pr\{\text{sample } i-j \text{ SR on } A_1 \text{ + guess } A_0\}$$

$$= \underbrace{\Pr\{\text{sample } i-j \text{ SR on } A_1\}}_{= y_{ij}}$$

Let $x^*_{ij} = x_{s-i,\, s-j}$

vX^* = upward transition probabilities corresponding to $A_0 E_1$

$$x^*_{ij} = \begin{cases} \sum_{k=0}^{i} \binom{i}{k}\binom{s-i}{j-i} t^{j-i+k}(1-t)^{s-j+i-k} \frac{j-i}{j-i+k} & \text{if } j \geq i \\ 0 & \text{if } j \leq i \end{cases}$$

Similarly, let

$y^*_{ij} = y_{s-i,\, s-j}$

$(1-w)(Y^* - X^*)$ = upward transition probabilities corresponding to $A_1 E$.

$P = wX + vX^* + (1-v)(Y-X) + (1-w)(Y^*-X^*)$
$+ \underbrace{(v+w-1)(1-t)^s I}_{\text{no stimulus elements sampled}}$.

We want $P\gamma, \ldots, P^n\gamma$

Find $P\gamma = vt\xi + [1-(v+w)t]\gamma$

$\xi = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

Let $\delta = \gamma - \frac{v}{v+w}\xi$

$P\delta = P\gamma - \frac{v}{v+w}\xi = [1-(v+w)t]\gamma - \frac{v}{v+w}[1-(v+w)t]\xi$

$= [1-(v+w)t]\delta$

$\Rightarrow P^n\delta = [1-(v+w)t]^n \delta$

Since $|1-(v+w)t| < 1$, $P^n\delta \to 0$ as $n \to \infty$.

$\exists\ P^* = \lim_{n\to\infty} P^n$ since have single recurrent class

$P^*\delta = 0$.

$P^* = \begin{pmatrix} \pi \\ \vdots \\ \pi \end{pmatrix}$, $\pi = (\pi_0, \pi_1, \ldots \pi_s)$

π_i = prob. of state i in equil.

Hence

$$\pi\delta = \pi\left(\gamma - \frac{v}{v+w}\xi\right) = 0$$

or $\pi\gamma = \frac{v}{v+w}$ since $\pi\xi = 1$

Hence the limiting prob of response A_1 is $\frac{v}{v+w}$

$$P^n\gamma = \left[\frac{v}{v+w}\right]\gamma + \left[1-(v+w)\right]^n\delta$$

transient term

On the other hand, the limiting probability of action E_1 is

$$\frac{v}{v+w}(1-w) + \frac{w}{(v+w)}v = \frac{v}{v+w} = P(A_1)$$

Note according to this model the subject does not max the # of correct guesses. He brings about an equil where he guesses ~~E_1~~ A_0 with the same frequency of E_1.

Digression:

If $A \leftrightarrow$ transient class, i.e., $A^n \to 0$

$$\exists\ (I-A)^{-1} = I + A + A^2 + \cdots$$

$$(I-A)(I+A+\cdots+A^{n-1}) \equiv I - A^n$$

$I \cdot A^n \to I$ & I has determinant 1.

Hence, for suff large n, $I-A^n$ must have non-zero determinant. However, the determ of a product is the product of the determinants, so

$$|I-A| \neq 0 \Rightarrow (I-A)^{-1} \text{ exists.}$$

$$I + A + A^2 + \cdots + A^{n-1} = (I-A)^{-1}(I-A^n)$$

As $n \to \infty$, $A^n \to 0$ QED.

For $j = 1, \ldots, s$

$n_j \equiv$ # times the process is in state j

$$u_j^k = \begin{cases} 1 & \text{if in state } j \text{ after } k^{th} \text{ transition} \\ 0 & \text{otherwise} \end{cases}$$

$M_i(n_j)$ = mean # times in j given start in i

Then $\{M_i(n_j)\} = (I-Q)^{-1} \equiv N$ where this is an $s \times s$ matrix
$i, j \in$ transient class T

Proof: $n_j = \sum_{k=0}^{\infty} u_j^k$

$$\{M_i(n_j)\} = \{M_i(\sum_k u_j^k)\} = \{\sum_k M_i(u_j^k)\}$$

$$= \{\sum_k [(1-P_{ij}^k)\cdot 0 + P_{ij}^k \cdot 1]\}$$

$$= \sum_k \{P_{ij}^k\} = \sum_k Q^k \equiv N \qquad P^k = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & Q^k & \\ 0 & & & \end{array}\right)$$

Returning to learning model, let $v \equiv 0$

$\bar{\gamma} = \gamma$ with first element removed

$P^n \delta = [1-(v+w)t]^n \delta$, $\delta = \gamma$

$P\gamma = [1-wt]\gamma$; $Q\bar{\gamma} = (1-wt)\bar{\gamma}$

$(I-Q)\bar{\gamma} = wt\,\bar{\gamma}$ or $\frac{1}{wt}\,\bar{\gamma} = N\bar{\gamma}$

$M_i(n_1)\frac{1}{s} + M_i(n_2)\frac{2}{s} + \cdots + M_i(n_s)\times\frac{s}{s} = \frac{1}{wt}\,\frac{i}{s}$

= element of $N\bar{\gamma}$ = mean of total wrong responses starting in i.

If assume SR's distrib at random initially:

$$p_i^0 = \frac{1}{2^s}\binom{s}{i}$$

Then the total # of wrong responses is

$$p^0 N\bar{\gamma} = \frac{1}{wt}\,p^0\gamma = \frac{1}{2wt}$$

HW add to 1st p.

$m=0,1,2,\dots \; ; \; \pi_k = \sum \pi_{m+k-1} b_m$

Normal Processes & Covariance

Stationary processes (Parzen Chapter 3)

$\{x(t)\,;\,t\in T\}$ is a stochastic process with finite second moments. It has mean:

$$m(t) = E\{x(t)\}$$

and covariance kernel

$$K(s,t) \quad Cov[x(t), x(s)] = E\{x(t)x(s)\} - E\{x(t)\}E\{x(s)\}$$

Example: $x(t) = x_0 + vt$; x_0 & v are r.v.'s.

$$m(t) = E(x_0) + tE(v)$$

$$K(s,t) = E\{x_0^2\} + (s+t)E\{x_0 v\} + st\,E\{v^2\}.$$
$$- E\{x_0\}^2 - (s+t)E\{x_0\}E\{v\} - st\,E\{v\}^2$$

$$= var\{x_0\} + (s+t)Cov\{x_0, v\} + st\,var\{v\}$$

Defn: A continuous parameter stochastic process $\{X(t)\,;\,t\geq 0\}$ has independent increments if for any sequence $t_0 < t_1 < \dots < t_n$ the n random variables

$$x(t_1)-x(t_0), \quad x(t_2)-x(t_1), \quad \dots \quad , x(t_n)-x(t_{n-1})$$

are independent.

Defn: The process has stationary independent increments if in addition if

$$x(t_2+h) - x(t_1+h) \quad \text{and} \quad x(t_2)-x(t_1)$$

have same distribution for any $t_1, t_2, h \geq 0$.

$\{x(t), t \geq 0\}$ is a <u>Wiener process</u> if:

(a) $\{x(t);\, t \geq 0\}$ has stationary independent increments

(b) $x(t)$ is normal for any $t > 0$

(c) $E\{x(t)\} = 0$ for all $t > 0$

(d) $x(0) = 0$.

We can show $x(t) \sim N(0, \sigma^2 t)$

For $s < t$,

$$K(s,t) = \mathrm{Cov}\{x(s),\ x(t) - x(s) + x(s)\}$$

$$= \mathrm{Cov}\{\underbrace{x(s),\ x(t) - x(s)}_{0}\} + \mathrm{Cov}\{x(s), x(s)\}$$

$$= \mathrm{var}\{x(s)\}$$

<u>Stationary Stochastic Processes</u>:

Intuitively, a stationary process is one with time-independent distribution:

A stochastic process $x(t)$, $t \in T$ with linear index set T is <u>strictly stationary of order k</u> if for any k points $t_1, \dots, t_k \in T$ $\{x(t_1), \dots, x(t_k)\}$ and $\{x(t_1+h), \dots, x(t_k+h)\}$ are identically distributed.

It is <u>strictly stationary</u> if it is strictly stationary of order k for all $k > 0$.

A different defn of stationarity:

A stochastic process is said to be covariance stationary if $K(s,t) = K(|s-t|)$ and is finite:
i.e., $\exists \; R(\nu) \ni R(\nu) = Cov[x(t), x(t+\nu)]$

Ergodicity:

~~Consider~~ A sequence of sample means of a discrete parameter stochastic process $x(t)$ ~~the sequence~~ is ergodic if

$$\lim_{T\to\infty} Var\{M_T\} = 0 \quad \text{where } M_T = \frac{1}{T}\sum_{t=0}^{T} x(t)$$

i.e., if $E\{M_T\} \approx \frac{1}{T}\sum_{t=1}^{T} x(t)$

for almost all sample realizations.

Theorem: ~~A~~
If the cov kernel of a discrete s.p. is bounded, the sample means are ergodic $\Leftrightarrow$

$$\lim_{t\to\infty} C(t) = 0$$

where $C(t) = Cov\{x(t), M_t\} = \frac{1}{t}\sum_{s=1}^{t} K(s,t)$

Note: If the process is cov-stationary, $\exists\; R(\nu)$, and

$$C(t) = \frac{1}{t}\sum_{s=1}^{t} R(t-s) = \frac{1}{t}\sum_{\nu=0}^{t-1} R(\nu)$$

Hence, if $\lim_{\tau \to \infty} R(\tau) = 0$, then $\lim_{t \to \infty} C(t) = 0$.

Convergence

(1) With probability one
(2) in probability
(3) in mean square
(4) in distribution

$$\left[\begin{array}{l} y(t) = \int_a^b x(t)\,dt \rightarrow E\{y(t)\} = \int_a^b E[m(t)]\,dt \\ Cov\{y(t), y(s)\} = \int_a^b \int_c^d K(s,t)\,ds\,dt \end{array}\right.$$

$$\left[\begin{array}{l} z(t) = \frac{d}{dt} x(t) \rightarrow E\{z(t)\} = \frac{d\,m(t)}{dt} \\ Cov\{z(t), z(s)\} = \frac{\partial^2}{\partial s \partial t} K(s,t) \end{array}\right.$$

Normal Processes:

$\{x(t)\}$ is a normal process if the r.v.'s $x(t_1), \ldots, x(t_n)$ are jointly normal for any $(t_1, \ldots, t_n) \in T$

Theorem: If $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ are jointly normal r.v.'s, then $\underline{y} = \underline{A}\,\underline{x}$ are jointly normal and

$$E\{\underline{y}\} = \underline{A}\, E\{\underline{x}\}$$

$$Cov\{y_i, y_j\} = \sum_{s,t} a_{js} a_{kt} Cov\{x_s, x_t\}$$

Theorem: If $\{z_n\}_{n=1}^{\infty} \to z$ in mean square, then z is normally distributed if the z_n are normal. In particular, if $x(t)$ is normal then

$$\int_0^t x(s)\,ds \text{ is normal if it exists.}$$

Filtered Poisson Process:

Calls are made in a telephone system with infinite capacity at instants $\tau_1 < \tau_2 < \dots$ The arrivals are Poisson (λ). The holding times $y_1, y_2, \dots$ are i.i.d.

$x(t)$ = # busy channels at time t

$$x(t) = \sum_{n=1}^{N(t)} w_0(t-\tau_n, y_n)$$

where $w_0(s,y)$ is defined for $y>0$ by

$$w_0(s,y) = \begin{cases} 1 & \text{if } 0 \le s \le y \\ 0 & \text{if } s<0 \text{ or } s>y \end{cases}$$

$$w_0(t-\tau_n, y_n) = \begin{cases} 1 & \text{if } \tau_n \le t \le \tau_n + y_n \\ 0 & \text{otherwise} \end{cases}$$

$x(t)$ is called a filtered Poisson process

Other applications:
- # claims outstanding in insurance co.
- # machines down

Theorem: If $x(t)$ is filtered Poisson, then

$$\varphi_{x(t)}(u) = \exp\left\{\lambda \int_0^t E\left[e^{iuw_0(t-\tau,y)} - 1\right] d\tau\right\}$$

$$E\{x(t)\} = \lambda \int_0^t E[w(t-\tau, y)]\,d\tau$$

$$\mathrm{var}[x(t)] = \lambda \int_0^t E[w_0^2(t-\tau, y)]\,d\tau$$

$$\mathrm{cov}[x(t_1), x(t_2)] = \lambda \int_0^{\min(t_1,t_2)} E\{w_0(t_1-\tau, y)\cdot w_0(t_2-\tau, y)\}\,d\tau$$

A more general function $w(s, y)$ leads to a more general filtered Poisson

Using $\varphi_{x(t)}(u) = \exp\{(e^{iu}-1)\lambda \int_0^t [1-F_y(s)]\,ds$

This process is Poisson with mean: $\lambda \int_0^t [1-F_y(s)]\,ds$

As $t \to \infty$, $x(t) \to$ Poisson with mean $\lambda E(y)$

$$\rho \equiv \lambda E(y) = \frac{\text{mean service time}}{\text{mean arrival time}}$$

$$P\{x(t)=0\} = e^{-\rho}$$

$$\mathrm{Cov} = \lambda \int_0^s E\{w_0(s-\tau, y)\, w_0(t-\tau, y)\}\,d\tau \qquad s \le t$$

$$= \lambda \int_0^s [1-F_y(t-s+u)]\,du \qquad \text{by } u = s-\tau$$

For exponential service rate μ,

$$\mathrm{Cov} = \frac{\lambda}{\mu}\{e^{-\mu(t-s)} - e^{-\mu t}\} \qquad s \le t$$

$$t = s+v \Rightarrow \mathrm{Cov} = \frac{\lambda}{\mu}\{e^{-\mu v}\}\{1-e^{-\mu s}\}$$

$$\lim_{\substack{s\to\infty \\ s \le t}} \mathrm{Cov}\{x(s), x(t)\} = \frac{\lambda}{\mu} e^{-\mu v} = R(v)$$

Conclusion: In equilibrium, we have a cov-stationary r.p. with mean $E\{x(t)\} = \lambda/\mu$ and cov fn:

$$R(v) = \frac{\lambda}{\mu} e^{-\mu|v|}$$

Note $\lim_{\tau\to\infty} R(\tau) = 0 \Rightarrow \mathrm{Var}\{M_T\} \to 0$ as $T \to \infty$

so $M_T \to E\{x(t)\} \to \frac{\lambda}{\mu}$

For a general service distribution,

$\lim_{s\to\infty} \mathrm{Cov}(s, s+\tau) = \lambda E(y)\left\{1 - \int_0^{\tau} \frac{1-F_y(u)}{E(y)}\,du\right\} \Rightarrow$ Cov-stationary

Confidence statements on realizations:

Observe a realization $x(t)$ with exponential service.
In equilibrium,

$$M_T = E\{x(t)\} = \frac{\lambda}{\mu}$$

$$K(s,t) = \frac{\lambda}{\mu} e^{-\mu|t-s|}$$

$$\mathrm{Var}\{M_T\} = \frac{1}{T^2}\int_0^T\int_0^T K(s,t)\,ds\,dt$$

$$= \frac{2}{T^2}\int_0^T dt \int_0^t ds\, K(s,t) = \frac{2\lambda}{\mu T^2}\int_0^T dt\int_0^t ds\, e^{-\mu(t-s)}$$

$$= \frac{2\lambda}{\mu^2 T} + \frac{2\lambda}{\mu^3 T^2}\left(1 - e^{-T\mu}\right)$$

Given: $\left.\begin{matrix}\lambda \leq 2000 \\ \mu \geq 10\end{matrix}\right\}$ How large should T be $\ni M_T$ differs from $E\{M_T\}$ by not more than 5 with 0.99 confidence?

Chebychev's Inequality:

$$\Pr\{|M_T - E(M_T)| \geq h\,\mathrm{var}(M_T)\} \leq \frac{1}{h^2}$$

We require: $\frac{1}{h^2} = \frac{1}{100}$ or $h = 10$

and $10\,\mathrm{var}(M_T) = 5$.

We guess that T is sufficiently large that $e^{-\mu T} = 0$, so

$$5 = 10\left[\frac{2\lambda}{\mu^2 T} + \frac{2\lambda}{\mu T^2}\right] \quad \text{or} \quad T^2 - \frac{4\lambda}{\mu^2} T - \frac{4\lambda}{\mu^3} = 0$$

$$\text{or} \quad T = \frac{4\lambda}{2\mu^2} \pm \frac{1}{2}\sqrt{\frac{16\lambda^2}{\mu^4} + \frac{16\lambda}{\mu^3}} = \frac{2\lambda}{\mu^2} \pm \frac{2\lambda}{\mu^2}\sqrt{1 + \frac{\mu}{\lambda}}$$

$$= \frac{2\lambda}{\mu^2}\left[1 + \sqrt{1 + \frac{\mu}{\lambda}}\right] \approx \frac{4\lambda}{\mu^2}$$

$$\text{or} \quad T \leq 80$$

<u>So</u>: observe for 80 hrs, take sample mean: Prob that this lies within 5 of true mean is at least .99

<u>Queue disciplines other than FIFO</u>:

1. Priority discipline
   (a) Non-preempt
   (b) Preempt
      (1) preempt resume
      (2) preempt repeat
2. LIFO
3. Other

<u>Non-preemptive Priority Queueing</u>

Ref: Cox & Smith pp 76–90.

Each customer in a priority class $j = 1, \ldots, K$. j-customers arrive Poisson with rate λ_j independent of other classes. Choose time scale so $\Sigma \lambda_j = 1$; hence all arrivals are Poisson with rate 1. Service times are independent with d.f. $F_j(x)$; the overall service time is distrib as

$$F(x) = \Sigma \lambda_j F_j(x)$$

Let m_j and v_j be the 1st & 2d moments of F_j; then the 1st & 2d moments of F are

$$m = \sum \lambda_j m_j \quad \text{and} \quad v = \sum \lambda_j v_j$$

Overall service rate $= \mu = \frac{1}{m}$

$$\rho = \frac{\lambda}{\mu} = \frac{\sum \lambda_j}{1/m} = m = \sum \lambda_j m_j$$

Defn:

(a) An epoch is the instant at which a service is completed.

(b) A j-epoch is the instant at which a j-customer begins service. This is called the event R_j

(c) If no customer is waiting when service is complete then a 0-epoch occurs & R_0 occurs

π_j = equilibrium probability that R_j occurs

Mean queuing time for j-customers

Fix $j \geq 2$ and let all ~~the~~ classes $1, \dots, j-1$ be lumped into a class H. H-customers lose priority within the new class & are served FIFO, but continue to have priority over classes $j, j+1, \dots, k$. We haven't affected the j-epochs by this lumping.

The expected number of 2-customers in the queue at a 2-epoch is:

$$1 + \frac{\frac{1}{2}\lambda_2 v}{\rho(1-\lambda_1 m_1)(1-\lambda_1 m_1 - \lambda_2 m_2)}$$

If $\lambda_1 \to 0$, then 2-customers effectively have top priority. In that case, the expected # of 2-customers is

$$1 + \frac{\frac{1}{2}\lambda_2 v}{\rho(1-\lambda_2 m_2)}$$

so the expected # of 1-customers at a 1-epoch is

$$1 + \frac{\frac{1}{2}\lambda_1 v}{\rho(1-\lambda_1 m_1)}$$

If we lump classes $1, \dots, j-1$ into a single class, H, j-customers become effectively 2-customers:

$$n_j = 1 + \frac{\frac{1}{2}\lambda_j v}{\rho(1-\lambda_H m_H)(1-\lambda_H m_H - \lambda_j m_j)}$$

where $\lambda_H m_H = \lambda_1 m_1 + \cdots + \lambda_{j-1} m_{j-1}$

To see, note $\lambda_H = \lambda_1 + \cdots + \lambda_{j-1}$; hence

$$F_H(x) = \frac{1}{\lambda_H}\sum_{i=1}^{j-1} \lambda_i F_i(x) \quad \text{so } m_H = \frac{1}{\lambda_H}\sum \lambda_i m_i \quad .$$

If we lump all customers into one class, π_0 is not different from the M/G/1 case: $\underline{\pi_0 = 1-\rho}$; (Use imbedded Markov chain to get this.)

A j-epoch occurs as the result of a j customer, & λ_j is the proportion of arrivals which are j. So

$$\pi_j = \lambda_j(1-\pi_0) = \lambda_j \rho$$

Calculation of the stationary distribution of n_2 suffices (by above argument)

<u>Calculation of distribution of n_2.</u>

T = arbitrary epoch & T' = next epoch.

q_1, q_2 = # of 1 & 2 customers at T &

q_1' & q_2' = # - - - - - - - T'

If T is a j-epoch, then $T'-T$ is the service time of a j customer.

ξ_{j1} & ξ_{j2} = # of 1 & 2 customers arriving between T & T'

$$E\left[z_1^{\xi_{j1}} z_2^{\xi_{j2}}\right] = F_j^*(u_{12}) \text{ where } u_{12} = \lambda_1(1-z_1) + \lambda_2(1-z_2)$$

If T is a <u>1-epoch</u>,

$$q_1' = q_1 + \xi_{11} - 1$$

$$q_2' = q_2 + \xi_{12}$$

If T is a <u>2-epoch</u>,

$$q_1' = \xi_{21}$$

$$q_2' = q_2 + \xi_{22} - 1$$

If T is a <u>j-epoch</u>, $j > 2$

$$q_1' = \xi_{j1}$$

$$q_2' = \xi_{j2}$$

If T is a <u>0-epoch</u>,

$$\left.\begin{array}{l} q_1' = \xi_{j1} \\ q_2' = \xi_{j2} \end{array}\right\} \text{ w/prob } \lambda_j \quad \text{for } j = 1, 2, \ldots, K$$

Combine these formulas to evaluate the generating fn

$$E[z_1^{q_1'} z_2^{q_2'}] = E\{z_1^{q_1+\xi_{11}-1} z_2^{q_2+\xi_{12}} \mid R_1 \text{ at } T\}\pi_1$$
$$+ E\{z_1^{\xi_{21}} z_2^{q_2+\xi_{22}-1} \mid R_2 \text{ at } T\}\pi_2$$
$$+\sum_{j=3}^{K} E\{z_1^{\xi_{j1}} z_2^{\xi_{j2}} \mid R_j \text{ at } T\}\pi_j$$
$$+ \pi_0 \sum_{j=1}^{K} \lambda_j E\{z_1^{\xi_{j1}} z_2^{\xi_{j2}} \mid R_0 \text{ at } T\}$$

These terms simplify (q's & ξ's independent), e.g.

$$E\{z_1^{q_1+\xi_{11}-1} z_2^{q_2+\xi_{12}} / R_1 \text{ at } T\} = \frac{1}{z_1} E\{z_1^{q_1} z_2^{q_2}\} E\{z_1^{\xi_{11}} z_2^{\xi_{12}}\} \mid R_1 \text{ at } T$$
$$= \frac{1}{z_1} G_1(z_1, z_2) F_1^*(u_{12}) \text{ where } G_1(z_1,z_2) = E\{z_1^{q_1} z_2^{q_2} \mid R_1 \text{ at } T\}$$

Finally, since (q_1', q_2') has same dist as (q_1, q_2)

$$\Rightarrow \quad E[z_1^{q_1'} z_2^{q_2'}] = \pi_1 G_1(z_1,z_2) + \pi_2 G_2(z_2) + 1 - \pi_1 - \pi_2 \quad \Leftarrow$$
$$G_1(z_1,z_2) = E\{z_1^{q_1} z_2^{q_2} \mid R_1 \text{ at } T\}$$
$$G_2(z_2) = E\{z_2^{q_2} \mid R_2 \text{ at } T\}$$

Equating this to the simplification of the above formula,

$$\begin{cases} \pi_1 G_1(z_1,z_2)\left[1 - \frac{F_1^*(u_{12})}{z_1}\right] + \pi_2 G_2(z_2)\left[1 - \frac{F_2^*(u_{12})}{z_2}\right] \\ + \sum_{j=3}^{K} \pi_j \left[1 - F_j^*(u_{12})\right] + \pi_0\left[1 - \sum_{j=1}^{K} \lambda_j F_j^*(u_{12})\right] = 0 \\ \text{where } \quad u_{12} = \lambda_1(1-z_1) + \lambda_2(1-z_2) \end{cases}$$

Differentiating re z_j and setting $z_1 = z_2 = 1$
gives another equation ~~in z~~ showing $\pi_j = \lambda_j \rho$

Second partials re: z_1 & z_2 gives 2 more equations
Solving the original & 2 new eqns with $z_1 = z_2 = 1$ gives

$$(1-\lambda_1 m_1) m_{11} = 1 - \lambda_1 m_1 + \frac{\lambda_1 v}{2\rho}$$

$$(1-\lambda_2 m_2) m_{22} = 1 - \lambda_2 m_2 + \lambda_1 m_1 m_{12} + \frac{\lambda_2 v}{2\rho}$$

$$(1-\lambda_1 m_1)\frac{m_{12}}{\lambda_2} = m_1 m_{11} + m_2 m_{22} + \frac{v}{\rho} - m_1 - m_2$$

where $m_{11} = \frac{\partial}{\partial z_1} G_1(z_1, z_2)\Big|_{z_1=z_2=1}$

$m_{22} = \frac{\partial}{\partial z_2} G_2(z_2)\Big|_{z_2=1} = E\{q_2 \mid R_2 \text{ at } T\}$

$m_{12} = \frac{\partial}{\partial z_2} G_1(z_1, z_2)\Big|_{1,1}$

Solve for m_{11}, m_{12}, and m_{22} and get

$$m_{22} = 1 + \frac{\lambda_2 v}{2\rho(1-\lambda_1 m_1)(1-\lambda_2 m_2 - \lambda_1 m_1)}$$

As in previous discussion, combine $1, \dots, j-1$ into H
Then

$$\Rightarrow \quad m_{jj} = 1 + \frac{\frac{1}{2}\lambda_j v}{\rho(1-\lambda_H m_H)(1-\lambda_H m_H - \lambda_j m_j)} \quad \Leftarrow$$

One fewer than m_{jj} is the exp # of j-cust to arrive during the queueing time of the leading j-cust, given that the j-cust does not go directly into service.

$$x_j = m_{jj} - 1$$

$Q_j \equiv$ mean queuing time of a j-customer

$= \frac{m_{jj}-1}{\lambda_j}\rho$ since $E(x_j) = \int_0^\infty E(x_j \mid T_j = t)\, dF_{T_j}(t)$

~~x_j = # j cust~~ $= \lambda_j \int_0^\infty t F_{T_j}(t) = \lambda_j Q$

Now impose a cost structure

$w_j \equiv$ cost per unit time of j-customer waiting

$C = \sum_{j=1}^{K} \lambda_j w_j Q_j$ = expected cost per unit time (note $\sum \lambda_j = 1$)

$$= \frac{v}{2} \sum_{j=1}^{K} \frac{\lambda_j w_j}{\left[1-\sum_{i=1}^{j-1}\lambda_i m_i\right]\left[1-\sum_{i=1}^{j}\lambda_i m_i\right]}$$

If $K > 3$ and we switch priority of 2 & 3-customers, giving a new cost C'

$$C - C' = \frac{v\delta}{2}\left(\frac{w_2}{m_2} - \frac{w_3}{m_3}\right), \quad \delta < 0$$

$$\Rightarrow C \le C' \Leftrightarrow \frac{w_2}{m_2} \ge \frac{w_3}{m_3}$$

So, minimize C by choosing priorities in order of descending $\frac{w_i}{m_i}$

Note if w_i all =, then choose in order of expected svc time (shortest first)

Suppose we can predict service times & arrivals have density fn $f(x)$ over service times.

By analogy to $Q = \sum \lambda_j Q_j = \frac{v}{2} \sum_j \frac{\lambda_j}{\left(1-\sum_{i=1}^{j-1}\lambda_i m_i\right)\left(1-\sum_{i=1}^{j}\lambda_i m_i\right)}$

$$Q_{min} = \frac{v}{2}\int_0^\infty \frac{f(x)\,dx}{\left[1-\int_0^x y f(y)\,dy\right]^2}$$

Stochastic Programming:

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Stochastic LP: $\max\ cx$
$$Ax = b$$
$$x \geq 0$$

Given uncertainty in the parameters, we can
(i) use sensitivity analysis
(ii) treat as r.v.'s & solve (hard) or use certainty equivalent.

Multi-stage problems:

$$\min E(c) = E\left(\phi(x_1, x_2 \mid E_2)\right)$$
$$A_{11}x_1 = b_1$$
$$A_{21}x_1 + A_{22}x_2 = b_2$$
$$x_1, x_2 \geq 0$$

where A_{ij} are known & b_1 is known
b_2 is unknown, depending on parameters in E_2 which are known only after x_1 is chosen.

(1) $x_1 \geq 0$ is chosen to satisfy $b = A_{11} x_1$

(2) E_2 occurs (randomly) & determines b_2

(3) $x_2 \geq 0$ is chosen to satisfy $A_{22}x_2 = b_2 - A_{21}x_1$

If $\phi(x_1, x_2 \mid E_2)$ is convex over allowed x_1 & x_2, then $\phi_0(x_1) = E_{E_2}\{ \inf_{x_2} \phi(x_1, x_2 \mid E_2)\}$ is convex & $x_1 = x_1^*$ solves the uncertainty problem if $\phi_0(x_1^*) = \min_{x_1} \phi_0(x_1)$.

Outline since Markov

1. Markov chains
   - Recurrence classes
   - Transient classes
   - Basic limit theorems & stationary probabilities
   - Absorption probabilities

2. Imbedded Markov chains – M/G/1

3. Capacity expansion & probabilistic growth

omit 4. Markov chains & learning theory

omit 5. Normal processes & covariance-stationary processes

6. Filtered Poisson processes

7. Priority queues.